Problem 5 Let $G = (V, E)$ be a graph on n vertices. Denote by $t(G)$ for the number of triangles of G. For an edge $e \in E$, write $t(e)$ the number of triangles it belongs to.

a.- Show that if the end-vertices of e are x and y, then $d(x) + d(y) - t(e) \leq n$, and $\sum_{e=\overline{xy}\in E}(d(x) + d(y)) - \sum_{e\in E} t(e) \leq n|E|$ (0.5 points).

b.- Conclude that

$$t(G) \geq \frac{|E|}{3n}(4|E| - n^2).$$

(*Hint:* use the same arguments as in the proof of Mantel's theorem given in the lectures, based on an application of Cauchy - Schwarz inequality) (1 point).

c.- Show that the previous relation implies that a graph on n vertices, n even, and $\frac{n^2}{4} + 1$ edges contains at least a linear number of triangles (0.5 points).

Solution:

a.- For a fixed edge $e = \overline{xy}$, $t(e)$ is the number of vertices which are incident, at the same time, to x and y. Hence, the value $d(x) + d(y) - t(e)$ counts the number of vertices which are incident with either x or y, *without repetition*. As this number is not bigger than the total number of vertices (n), we conclude that this value is $\leq n$.

The second expression is obtained by summing over all edges, and observing that $\sum_{e\in E} n = n|E|$.

b.- We start from the expression $\sum_{e=\overline{xy}\in E}(d(x) + d(y)) - \sum_{e\in E} t(e) \leq n|E|$ deduced in $a.-$. As we showed in the lectures, $\sum_{e=\overline{xy}\in E}(d(x) + d(y)) = \sum_{x\in V} d(x)^2$, and then we can apply the Cauchy-Schwartz inequality in order to get that

$$\sum_{x\in V} d(x)^2 \geq \frac{1}{n}\left(\sum_{x\in V} d(x)\right)^2 = \frac{4}{n}|E|^2.$$

Observe also that each triangle in G is defined by 3 edges. Hence, $\sum_{e\in E} t(e) = 3t(G)$. Putting all the thing together we obtain that:

$$\frac{4}{n}|E|^2 - 3t(G) \leq n|E|$$

and isolating $t(G)$ we obtain the inequality as claimed.

c.- We substitute $|E| = \frac{n^2}{4} + 1$ in the expression deduced in part $b.-$. We obtain that:

$$t(G) \geq \frac{|E|}{3n}(4|E| - n^2) = \frac{\frac{n^2}{4} + 1}{3n}\left(4\left(\frac{n^2}{4} + 1\right) - n^2\right) = \frac{n^2 + 4}{3n} = \frac{1}{3}n + \frac{4}{3n}.$$

Hence the number of triangles in the graph must be at least linear.