

Measure Theory and the Lebesgue Integral

Bottle: The Element of Integration
and Lebesgue Measure

Introduction: from Riemann to Lebesgue

Recall that given a set $E \subseteq \mathbb{R}$, the indicator function of E , that we write $\mathbb{I}_E(x)$ is defined as

$$\mathbb{I}_E(x) = \begin{cases} 0 & x \notin E, \\ 1 & x \in E. \end{cases}$$

when considering a set of intervals $\{E_i\}_{i=1}^n$, we can sum the corresponding indicator functions:

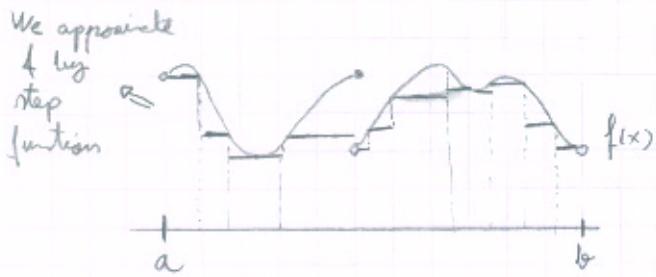
Def / An step function (or staircase function) is a function of the form

$$\varphi = \sum_{i=1}^m c_i \mathbb{I}_{E_i}, \quad \{E_i\}_{i=1}^m \text{ is a set of disjoint intervals and } c_i \in \mathbb{R}.$$

The integral of an step function is easily computed: if the length of E_i is b_i , then we write

$$\int_{-\infty}^{\infty} \varphi(x) dx = \int_{-\infty}^{\infty} \sum_{i=1}^m c_i \mathbb{I}_{E_i}(x) dx = \sum_{i=1}^m c_i \cdot b_i$$

How we define now the Riemann integral? Assume that $f: [a, b] \rightarrow \mathbb{R}$ is a (bounded) function with "not too many discontinuities".



We define two special parameters:

$$L(f; [a, b]) = \sup \left\{ \int_{-\infty}^{\infty} \varphi(x) dx \mid \varphi \text{ is step function, } \varphi(x) \leq f(x) \right\}$$

$$U(f; [a, b]) = \inf \left\{ \int_{-\infty}^{\infty} \varphi(x) dx \mid \varphi \text{ is step function, } \varphi(x) \geq f(x) \right\}$$

Def / We then say that f is Riemann integrable if $L(f; [a, b]) = U(f; [a, b])$.

Q: Are there "natural" functions that are NOT Riemann integrable? \Rightarrow Of course!

Ex / Consider $E = \mathbb{Q} \cap [0, 1]$, and let $f = \mathbb{I}_E$. We show that this function is NOT Riemann integrable. This is true because $L(f; [0, 1]) = 0$, but $U(f; [0, 1]) = 1$ (this is true because between each pair of real numbers there exist a rational and an irrational number).

So, our task now is to generalize the Riemann integral in order to be able to extend the notion of integrability. In particular, the main point is to extend the notion of measure of a set: we already know that the measure of an interval is its length. We want that the measure of E in the previous example is 0, then $\int f = 0$.

Measurable functions and σ -algebras.

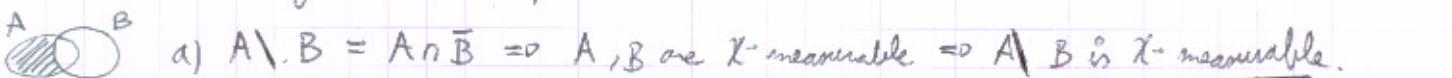
The first definition is the notion of σ -algebra:

Def/ let X be a set. A family of subsets $\mathcal{X} \subseteq P(X)$ is a σ -algebra if it satisfies the following properties:

- (σ 1) $\emptyset \in \mathcal{X}$.
- (σ 2) if $A \in \mathcal{X}$, then $A^c = \bar{A} = X - A \in \mathcal{X}$
- (σ 3) if $A_n \in \mathcal{X} \forall n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{X}$.

Def/ let X be a set and \mathcal{X} a σ -algebra over X . Then we say that (X, \mathcal{X}) is a measurable space and the elements of \mathcal{X} are X -measurable sets.

Obs/ Some easy constructions fit into this context:

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- a) $A \setminus B = A \cap \bar{B} \Rightarrow A, B$ are X -measurable $\Rightarrow A \setminus B$ is X -measurable.
 - b) If $\{A_n\}_{n=1}^{\infty}$ is a sequence of X -measurable sets, then $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \bar{A}_n$. Hence, the countable intersection of measurable sets is measurable.

let us see some examples:

- ① Every set X and $P(X) = \mathcal{X}$ defines the measurable space $(X, P(X))$
- ② Every set X and $\mathcal{X} = \{\emptyset, X\}$
- ③ Every set X , $E \subset X$ and $\mathcal{X} = \{\emptyset, X, E, \bar{E}\}$.
- ④ Given 2 σ -algebras $\mathcal{X}_1, \mathcal{X}_2$ over X , then $\mathcal{X}_3 = \mathcal{X}_2 \cap \mathcal{X}_1$ is also a σ -algebra over X .
- ⑤ Given X and $Z \subseteq P(X)$, $P(X)$ is a σ -algebra containing (in the sense of set inclusion) Z . From ④ 2 σ -algebras containing Z it also contains Z and it is a σ -algebra.

Hence, there is a σ -algebra which contains Z and it is contained in all σ -algebras containing Z . We call it the σ -algebra generated by Z . (we write it $C(Z)$)

- ⑥ For $X = \mathbb{R}$, we consider $Z = \{(a, b) \subseteq \mathbb{R}\}$. Then $C(Z)$ is called Borel algebra. We write $C(Z) = \mathcal{B}$, and a $B \in \mathcal{B}$ is a borelian (or Borel set)
- ⑦ let $X = \mathbb{R}^* \cup \{-\infty, +\infty\}$. If E is a borelian, then we write $E_1 = E \cup \{-\infty\}$, $E_2 = E \cup \{+\infty\}$ and $E_3 = E \cup \{-\infty, +\infty\}$. Finally,

$$\mathcal{B}^* = \bigcup_{E \in \mathcal{B}} \{E, E_1, E_2, E_3\} \text{ is called the } \underline{\text{extended Borel algebra}} \text{ (on } \mathbb{R}^*)$$

We can now define functions over measurable spaces:

Def/ let (X, \mathcal{X}) be a measurable set, and $f: X \rightarrow \mathbb{R}$ a function. f is said to be X -measurable if $A_q = \{x \in X : f(x) > q\} \in \mathcal{X}$.

Equivalently, f is X -measurable iff $f^{-1}((q, +\infty)) \in \mathcal{X}$ for all $q \in \mathbb{R}$. let us see that this definition is equivalent to different ways to take the antimage.

Prop/ let (X, \mathcal{X}) a measurable space, $f: X \rightarrow \mathbb{R}$ a function. It is equivalent that:

- a) $\forall q \in \mathbb{R}, A_q = f^{-1}((q, +\infty)) \in \mathcal{X}$
- b) $\forall q \in \mathbb{R}, B_q = f^{-1}((-∞, q]) \in \mathcal{X}$
- c) $\forall q \in \mathbb{R}, C_q = f^{-1}([q, +\infty)) \in \mathcal{X}$
- d) $\forall q \in \mathbb{R}, D_q = f^{-1}((-∞, q)) \in \mathcal{X}$

Proof / As $\bar{A}_\alpha = B_\alpha$ and $\bar{C}_\alpha = D_\alpha$, then (a) \Leftrightarrow (b) and (c) \Leftrightarrow (d). Let us show that (a) \Leftrightarrow (c). Observe that

$$[\alpha, +\infty) = \bigcap_{n=1}^{\infty} (\alpha - \frac{1}{n}, +\infty); \quad (\alpha, +\infty) = \bigcup_{n=1}^{\infty} [\alpha + \frac{1}{n}, +\infty)$$

and, as $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, then

$$C_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha - \frac{1}{n}}, \quad A_\alpha = \bigcup_{n=1}^{\infty} C_{\alpha + \frac{1}{n}} \quad (\text{(c)} \Rightarrow \text{(a)})$$

$$\hookrightarrow \bar{C}_\alpha = \overline{\bigcap_{n=1}^{\infty} A_{\alpha - \frac{1}{n}}} = \bigcup_{n=1}^{\infty} \bar{A}_{\alpha - \frac{1}{n}} \quad (\text{(a)} \Rightarrow \text{(c)})$$

Indeed, all types of U, \cap !

Ex/

① $f: X \rightarrow \mathbb{R}$, $f(x) = c$ $\forall x \in X$ is measurable:

- if $\alpha \geq c$, then $\{x \in X : f(x) > \alpha\} = \emptyset \in \mathcal{X}$ ✓ { \Rightarrow Not depending on the σ -algebra! }
- if $\alpha < c$, then $\{x \in X : f(x) \leq \alpha\} = X \in \mathcal{X}$ ✓

② $E \in \mathcal{X}$, \mathbb{I}_E is X -measurable:

- if $\alpha \geq 1$, $\{x \in X : f(x) > \alpha\} = \emptyset \in \mathcal{X}$ ✓
- if $\alpha \in [0, 1]$, $\{x \in X : f(x) > \alpha\} = E \in \mathcal{X}$ ✓
- if $\alpha \in (-\infty, 0)$, $\{x \in X : f(x) > \alpha\} = X \in \mathcal{X}$ ✓

③ Every continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable: $f^{-1}((\alpha, +\infty))$ is an open set, and hence it is the numerable union of open interval sets (which is a Borel set!).

We can build a big variety of functions starting from given ones:

Prop / let $f, g: X \rightarrow \mathbb{R}$, (X, \mathcal{X}) a measurable space and f, g X -measurable. Then, if $c \in \mathbb{R}$, $c f$, f^2 , $f+g$, fg and $|f|$ are X -measurable. Also, $f^+ = f \vee 0$ and $f^- = (-f) \wedge 0$.

Proof/

$c f$: if $c=0$, then the function is constant. If $c > 0$, $\{x \in X : cf(x) > \alpha\} = \{x \in X : f(x) > \frac{\alpha}{c}\} \in \mathcal{X}$. Similarly if $c < 0$.

f^2 : if $\alpha < 0$, $\{x \in X : f(x)^2 > \alpha\} = X \in \mathcal{X}$. If $\alpha \geq 0$, then

$$\{x \in X : f(x)^2 > \alpha\} = \{x \in X : f(x) > \sqrt{\alpha}\} \cup \{x \in X : f(x) < -\sqrt{\alpha}\} \in \mathcal{X}.$$

$f+g$: this requires more work. Let $r \in \mathbb{Q}$, and $S_r = \{x \in X : f(x) > r\} \cap \{x \in X : g(x) > r\}$. Let us show that

$$\{x \in X : (f+g)(x) > r\} = \bigcup_{r \in \mathbb{Q}} S_r \quad \xrightarrow{\text{a3-property}}$$

The inclusion \supseteq is obvious. To see \subseteq , let $x \in X$ such that $(f+g)(x) > r$. Let $k = r - g(x)$. Then $\alpha < f(x) + g(x) = f(x) + r - k \Rightarrow f(x) > k$. Take now $r' \in \mathbb{Q}$ such that $k < r' < f(x)$. Then $r' < f(x)$ and $g(x) = r - k > r - r'$, and hence $x \in S_{r'}$.

fg : Use that $fg = \frac{1}{4} ((f+g)^2 - (f-g)^2)$

$|f|$: change the argument for f^2 .

$$f^+ \cup f^- \cdot \text{ write } f^+ = \frac{1}{2} (|f| + f), \quad f^- = (|f| - f) \frac{1}{2}$$

For convenience we are interested in functions $f: X \rightarrow \mathbb{R}^*$; we make a more precise study here:

Def / Let (X, \mathcal{X}) be a measurable space. We say that $f: X \rightarrow \mathbb{R}^*$ is X -measurable if $\forall a \in \mathbb{R}$,

$\{x \in X : f(x) > a\} \in \mathcal{X}$. The set of such functions is denoted by $H(X, \mathcal{X})$.

Ob / $A \in \mathcal{X} \iff \{x \in X : f(x) = +\infty\} \subseteq A = \bigcap \{x \in X : f(x) > a\}$.

Then, the previous results apply (with the comment that if $c=0$, then $c \cdot f = 0$) in the case of $f, f^2, |f|$ and f^+, f^- . For $f+g$ we need to do something else.

f+g is not defined in $E_1 = \{x \in X : f(x) = -\infty, g(x) = +\infty\}$, $E_2 = \{x \in X : f(x) = +\infty, g(x) = -\infty\}$ which are disjoint in $X = \varnothing$ we define $f+g = 0$ in $E_1 \cup E_2$

f₀: we need an auxiliary result (which is important later!)

Lemma / Let $\{f_n\}_{n \geq 1}$ be a sequence of functions in $H(X, \mathcal{X})$, and let $f(x) = \inf f_n(x)$, $F(x) = \sup f_n(x)$, $f^*(x) = \liminf_{n \rightarrow \infty} f_n(x)$, $F^*(x) = \limsup_{n \rightarrow \infty} f_n(x)$. Then f, F, f^* and $F^* \in H(X, \mathcal{X})$.

Proof / Observe that $\{x \in X : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x \in X : f_n(x) \geq a\}$ and $\{x \in X : F(x) \geq a\} = \bigcup_{n=1}^{\infty} \{x \in X : f_n(x) \geq a\}$, which establishes the result for f and F .

For f^* and F^* , observe that $f^*(x) = \sup \{ \inf_{n \geq m} f_n(x) \}$, $F^*(x) = \inf \{ \sup_{n \geq m} f_n(x) \}$, and applying the result for \inf^* and \sup^* we get the result.

Corollary / If $\{f_n\}_{n \geq 1}$ is a sequence of functions of $H(X, \mathcal{X})$ which converge pointwise to $f: X \rightarrow \mathbb{R}^*$, then $f \in H(X, \mathcal{X})$.

Proof / Just observe that $f(x) = \lim f_n(x) = \liminf f_n(x)$.

Let us prove now that if $f, g \in H(X, \mathcal{X})$, then $f \cdot g \in H(X, \mathcal{X})$. Define $f_n(x) = \begin{cases} 1 & \text{if } f(x) \leq n \\ 0 & \text{if } f(x) > n \end{cases}$, and similarly for g_n . In particular, $f_n \cdot g_m$ is X -measurable for all n, m , so $f_n \cdot g_m$ take values in \mathbb{R} . Hence,

$$f(x) \cdot g(x) = \lim_m (f(x) \cdot g_m(x)) = \lim_m (\lim_n f_n(x) g_m(x)) \Rightarrow \text{all are } X\text{-measurable}$$

by the previous corollary.

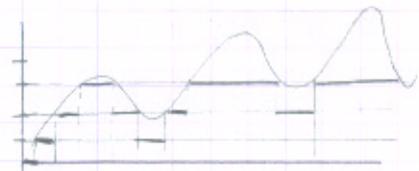
We finally see a proposition which will be important later:

Prop / Let $f \in H(X, \mathcal{X})$, $f(x) \geq 0 \quad \forall x \in X$. Then there exists a sequence $\{\varphi_n\}_{n \geq 1}$ of functions in $H(X, \mathcal{X})$ such that

(i) $0 \leq \varphi_n(x) \leq \varphi_{n+1}(x) \quad \forall n \in \mathbb{N}, \forall x \in X$

(ii) $\text{Im } \varphi_n$ is finite.

(iii) $f(x) = \lim \varphi_n(x)$ pointwise.



Proof / Take $m \in \mathbb{N}$. Define

$$E_{k,n} = \left\{ x \in X : k \cdot 2^{-n} \leq f(x) < (k+1) \cdot 2^{-n} \right\}, \quad k=0, 1, \dots, n \cdot 2^{-n}-1$$

$$E_{k,n} = \left\{ x \in X : f(x) \geq k \cdot 2^{-n}, \quad k = n \cdot 2^{-n} \right\}$$

$f(x) \geq 0!!$

This is obvious that all these sets are disjoint, all $\in X$ and that its union is X . Then we define $\varphi_n(x) = k \cdot 2^{-n}$ if $x \in E_{k,n}$. Again, for all $a \in \mathbb{R}$,

$$\{x \in X : \varphi_n(x) > a\} = \bigcup_{k=0, k \geq a}^{n \cdot 2^{-n}} E_{k,n} \in \mathcal{X} \Rightarrow \varphi_n \in H(X, \mathcal{X})$$

It is clear that $\varphi_n(x) \geq 0$ and that it takes a finite number of values, that $\varphi_n(x) \leq f(x)$ and $|f(x) - \varphi_n(x)| \leq 2^{-n} \Rightarrow f(x) = \varphi_n(x) \text{ a.e.}$ Let us only check that $\varphi_n(x) \leq \varphi_{n+1}(x)$.

When we go from n to $n+1$, we obtain twice the number of sets we had before (the ranks of each $E_{k,n}$ is 2^{-n}). Writing the indices "old" and "new" for the values n and $n+1$, then we have that

$$x \in E_{k,\text{old},n} \Rightarrow x \in E_{k,\text{new},(n+1)} \text{ with } k_{\text{new}} > k_{\text{old}} - 1 \Rightarrow k_{\text{new}} \geq 2k_{\text{old}}$$

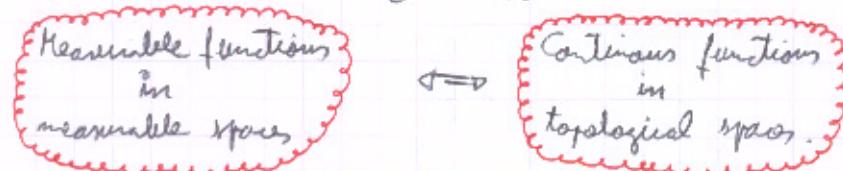
$$\ell_{n+1}(x) = k_{\text{new}} 2^{-(n+1)} \geq 2k_{\text{old}} 2^{-(n+1)} = k_{\text{old}} 2^{-n} = \ell_n(x)$$

And so $\ell_{n+1}(x) \geq \ell_n(x)$, $\forall x \in X$, $\forall n \in \mathbb{N}$.

A last definition that can clarify a little is the following

Def/ let (X, \mathcal{X}) , (Y, \mathcal{Y}) be measurable spaces. A function $f: X \rightarrow Y$ is measurable if $\forall E \in \mathcal{Y}$ $f^{-1}(E) \in \mathcal{X}$.

So, there is the following analogy:



Measures

Def/ Given a measurable space (X, \mathcal{X}) , a measure μ is a function $\mu: \mathcal{X} \rightarrow \mathbb{R}^+$ such that

$$(M1) \mu(\emptyset) = 0$$

$$(M2) \mu(E) \geq 0 \quad \forall E \in \mathcal{X}$$

$$(M3) \text{ Given } \{E_n\}_{n=1}^{\infty} \text{ a sequence in } \mathcal{X}, E_n \cap E_m = \emptyset \text{ for } n \neq m, \text{ then } \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Def/ If $\mu(E) \neq +\infty \quad \forall E \in \mathcal{X}$, we say that μ is a finite measure.

Def/ If there is a sequence $\{E_n\}_{n=1}^{\infty}$ such that $X = \bigcup_{n=1}^{\infty} E_n$ and $\mu(E_n) < \infty$, then μ is σ -finite.

Def/ A triplet (X, \mathcal{X}, μ) is called a measure space.

An important notion is the following

Def/ let (X, \mathcal{X}, μ) a measure space. We say that an statement is μ -almost always (μ -a.a.) if there is $N \in \mathcal{X}$ such that $\mu(N) = 0$ and the statement is true in N^c .

For instance, the sequence $\{f_n\}_{n=1}^{\infty}$ converge pointwise to f μ -a.a. if writing $E = \{x \in X : f_n(x) \neq f(x)\}$, then $\mu(E) = 0$. Let us see some examples:

Ex/ ① Take $X \neq \emptyset$, $\mathcal{X} = P(X)$. Define $\mu_1(E) = 0 \quad \forall E \in \mathcal{X}$, $\mu_2(\emptyset) = 0$, $\mu_2(E) = +\infty$ if $E \neq \emptyset$.

② Take (X, \mathcal{X}) , $X \neq \emptyset$, $p \in X$. We define $\mu_p(E) = \begin{cases} 0, & p \notin E \\ 1, & p \in E \end{cases}$. This is also a measure.

③ $X = \mathbb{N}$, $\mathcal{X} = P(X)$, $\mu(E) = |E|$. Observe that μ is not finite but σ -finite.

We will say more words later about \mathbb{R} . For the moment, let us see some properties:

Prop/ let (X, \mathcal{X}, μ) a measure space. If $E, F \in \mathcal{X}$, $E \subseteq F$, then $\mu(E) \leq \mu(F)$. Additionally,

Proof / As $F = E \cup (F \setminus E)$, and $E \cap (F \setminus E) = \emptyset$ and $F \setminus E \in X$ we have that $\mu(F) = \mu(F \setminus E) + \mu(E)$.
 As $\mu(F \setminus E) \geq 0$, then $\mu(F) \geq \mu(E)$ as desired. If $\mu(E) < +\infty$, then $\mu(F) - \mu(E) = \mu(F \setminus E)$.

Prop / Let (X, \mathcal{X}, μ) a measure space.

- i) If $\{E_n\}_{n=1}^{\infty}$ is an increasing sequence of sets, $E_n \in \mathcal{X}$, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n)$.
- ii) If $\{F_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets, $F_n \in \mathcal{X}$ / $\mu(F_1) < +\infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim_{n \rightarrow \infty} \mu(F_n).$$

Proof / We show i). If $\mu(E_n) = +\infty$ for some n , then both sides are equal to $+\infty$. Assume then that $\forall n, \mu(E_n) < +\infty$. Write $A_1 = E_1, A_m = E_m \Delta E_{m-1}$. Then

$$E_n = \bigcup_{m=1}^n A_m \Rightarrow \bigcup_{n=1}^{\infty} E_n = \bigcup_{m=1}^{\infty} A_m \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{m=1}^{\infty} A_m\right) = \sum_{m=1}^{\infty} \mu(A_m) = \lim_{M \rightarrow \infty} \sum_{m=1}^M \mu(A_m).$$

By the previous proposition, $\mu(F_n) = \mu(E_n) - \mu(E_{n-1})$ because $\mu(F_n), \mu(E_{n-1}) < +\infty$ for all n . Hence,

$$\lim_{m \rightarrow \infty} \mu(F_m) = \mu(E_m) = \lim_{m \rightarrow \infty} \mu(A_m) = \lim_{n \rightarrow \infty} \sum_{m=1}^n \mu(A_m) = \lim_{n \rightarrow \infty} \mu(E_n)$$

To prove ii), we use the sequence $E_n = F_1 \Delta F_n$, which is increasing.

Lebesgue measure over \mathbb{R}

The construction of a measure over \mathbb{R} which particularizes the length of an interval it is not immediate. We sketch the ideas or how to do that (Chapter 9 in Bartle gives all the details)

Starting point : we have a "measure" over intervals, but not ALL borel sets are build taking unions of intervals =D Is there a way to "extend" the length ?

Def / Let X be a set, $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra if

- (a1) $\emptyset \in \mathcal{A}$
- (a2) $E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$

(a3) E_1, \dots, E_r in a finite family of elements in $\mathcal{A} \Rightarrow \bigcup_{i=1}^r E_i \in \mathcal{A}$.

Def / Given an algebra \mathcal{A} over X , a function $\mu: \mathcal{A} \rightarrow \mathbb{R}^*$ is a measure if

$$(m1) \mu(\emptyset) = 0$$

$$(m2) \mu(E) \geq 0 \quad \forall E \in \mathcal{A}$$

(m3) If $\{E_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint sets of \mathcal{A} , such that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

Then it is easy to show that the family \mathcal{I} of all finite unions of intervals of the form $(a, b], (-\infty, b], (a, +\infty), (-\infty, +\infty)$ is an algebra, and $\ell = \text{length}$ is a measure over this algebra.

Extension : for $B \in \mathcal{P}(X)$ we define $\mu^*(B) = \inf \sum_{j=1}^{\infty} \mu(E_j)$, where $B \subseteq \bigcup_{j=1}^{\infty} E_j$. This is well defined because $B \subseteq \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{\infty} \dots$. This is called the exterior measure (outer measure)

Def / A set $E \subseteq X$ is μ^* -measurable if $\forall A \subseteq X, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$ (Lebesgue's additivity condition)

Def / The set of μ^* -measurable sets associated to the algebra \mathcal{A} is written by \mathcal{A}^* .