

The precise order of decay is given in the following theorem:

Theorem / let  $\mathbb{Z}_{\geq 0}$  be a branching process with initial offspring distribution  $X$  with  $\mu \leq 1$  and  $\sigma^2 < \infty$ . Then,

- i) If  $\mu < 1$ , then there is  $C_B \in (0, \infty)$  such that  $p(Z > n) \sim C_B \mu^n$  as  $n \rightarrow \infty$ .
- ii) If  $\mu = 1$ , then  $p(Z > n) \sim \sigma^2 n$  as  $n \rightarrow \infty$ .

Proof / We analyze only the case i), ii) could be done by the same means; assume that  $\mu = G'(1) < 1$ . The sequence  $G_n(0)$  tends to 1, so we need to measure how fast we go to the limit:

$$\begin{aligned} 1 - G_{n+1}(0) &= 1 - G(G_n(0)) = 1 - G(1 - (1 - G_n(0))) \\ &= 1 - (1 - G'(1)(1 - G_n(0))) + O((1 - G_n(0))^2) = \mu(1 - G_n(0)) + O((1 - G_n(0))^2) \end{aligned}$$

Next step is based on controlling the error term (observe that if  $G$  were linear, then we obtain that  $1 - G_n(0) = \mu^n$ , and we are done). Observe that the function  $G$  is convex, hence the second derivative of  $G$  is positive. As  $O((1 - G_n(0))^2) = -G''(\xi)/2(1 - G_n(0))^2$ , we have that

$$\mu(1 - G_n(0)) - C(1 - G_n(0))^2 \stackrel{\textcircled{1}}{\leq} 1 - G_{n+1}(0) \stackrel{\textcircled{2}}{\leq} \mu(1 - G_n(0)).$$

Let us study each inequality separately.

$$\textcircled{1} \quad 1 - G_{n+1}(0) \leq \mu(1 - G_n(0)) \leq \dots \leq \mu^{n+1}.$$

\textcircled{2} We manipulate the expression:

$$\begin{aligned} \mu(1 - G_n(0)) - C(1 - G_n(0))^2 &\leq 1 - G_{n+1}(0) \leq \mu(1 - G_n(0)) \\ 1 - C\mu^{n+1} &\leq 1 - \frac{C}{\mu}(1 - G_n(0)) \leq \frac{\mu^{n+1} (1 - G_{n+1}(0))}{\mu^n (1 - G_n(0))} \leq 1 \end{aligned}$$

So, we have that writing  $U_n = \frac{\mu^{n+1} (1 - G_{n+1}(0))}{\mu^n (1 - G_n(0))}$ , then  $U_n$  tends to 1 geometrically fast. Finally, we use the following well known result by Weierstrass:

Theorem / (Weierstrass) let  $a_n$  be a sequence of reals with  $0 < a_n < 1$ . Then the infinite product  $\prod_{n=1}^{\infty} (1-a_n)$  converges iff  $\sum_{n=1}^{\infty} a_n$  converges to a nonzero real.

We are under the assumptions of the theorem, hence, writing

$$\begin{aligned} \prod_{n=1}^{\infty} U_n &= C_B^{-1} = \prod_{n=1}^{\infty} \prod_{r=1}^n U_r = \prod_{n=1}^{\infty} U_1 U_2 \dots U_{n-1} = \prod_{n=1}^{\infty} \frac{\mu^n (1 - G_n(0))}{\mu^{n-1} (1 - G(0))} = \\ &= \prod_{n=1}^{\infty} \mu^n (1 - G_n(0)) = C_B^{-1} (1 - G(0)) \end{aligned}$$

$$\text{So, } \prod_{n=1}^{\infty} 1 - G_n(0) = L \quad p(Z > n) \sim C_B \mu^n.$$

Obs / The changes in the critical case are the following: now  $\mu = 1$ , and we have the following approximation:

$$1 - G_{n+1}(0) = 1 - G_n(0) - \frac{1}{2} \sigma^2 (1 - G_n(0))^2 + O((1 - G_n(0))^3)$$

so  $1 - G_{n+1}(0) \approx 1 - G_n(0) - \frac{1}{2} \sigma^2 (1 - G_n(0))^2$ . Now the trick is to see that if  $A_n$  satisfies the relation  $A_{n+1} = A_n - B A_n^2$ , then  $B_n = 1/A_n$  satisfies that  $B_{n+1} = B_n + b + \frac{1}{2} B_n^{-1}$ . (16)

## Random walks

In this section we study the following problem: consider  $\mathbb{R}^d$  and the 2d possible unit steps. At each step we decide, uniformly at random among the 2d steps, to make the next step.

Q: Can we say something about the return to 0?

Def / A random walk in dimension  $d$  is a sequence of independent random variables  $X_1, X_2, \dots$  with uniform distribution in the 2d possible unit steps in  $\mathbb{R}^d$ . The sequence  $S_0 = 0, S_n = \sum_{i=1}^n X_i$  is also called the random walk.

It is then clear that  $E[X_i] = 0$ , so  $E[S_n] = 0$ , and that  $E[S_n^2] = n$ . In this section we will study the event "the walk never returns to 0". This is an event because it can be written as:

$$\text{Escape} = \{\text{the walk never returns to 0}\} = \bigcap_{n \geq 0} \{S_n \neq 0\}$$

Def /  $p_{\text{esc}} = p(\text{walk never returns to 0})$ .

Def / A random walk is recurrent iff  $p_{\text{esc}} = 0$ ; a random walk is transient iff  $p_{\text{esc}} > 0$ .

The main result we show is Polya's theorem:

Theorem / (Polya's Theorem) A random walk in  $\mathbb{R}^d$  is recurrent for  $d=1, 2$ , and transient for  $d \geq 3$ .

Proof / We show the case  $d=1, 2, 3$ ; case  $d \geq 3$  are very similar to  $d=3$ . We use, however, some techniques from GFs in the general case. Write  $u_n = p(\text{walk starts at } 0 \text{ is at } 0 \text{ at the } n\text{th step})$ ,  $u_0 = 1$ , and  $e_n = 1$  if the walker is at 0 at time  $n$ . Then  $T = \sum_n e_n$  is the total number of times at 0. Writing then

$$m = E[T] = \sum_{n=0}^{\infty} E[e_n] = \sum_{n=0}^{\infty} u_n$$

Let us compute  $m$  in another way. Write  $F_n = \{S_n = 0\} \cap \{S_{n-1} \neq 0\}$ , and  $f_n = p(F_n)$ .

In particular,  $\{F_n\}$  is a sequence of disjoint events, and  $\text{Escape} = \bigcup F_i$ . Then, looking at the first return:

$$\begin{aligned} u_0 &= 1 \\ u_1 &= f_0 u_1 + f_1 u_0 \\ u_2 &= f_0 u_2 + f_1 u_1 + f_2 u_0 \\ u_3 &= f_0 u_3 + f_1 u_2 + f_2 u_1 + f_3 u_0 \end{aligned} \quad \left. \begin{aligned} U(s) &= \sum u_n s^n, F(s) = \sum f_n s^n \\ \Rightarrow & \\ U(s) - 1 &= U(s) F(s) \Rightarrow F(s) = 1 - \frac{1}{U(s)} \end{aligned} \right.$$

so,  $F(1) = 1 - \frac{1}{m} = \sum_{n=1}^{\infty} f_n$ . In other words,  $p_{\text{esc}} = F(1) = 1 - \frac{1}{m}$ , so we have the following dichotomy:

i)  $m = \infty \Rightarrow F(1) = 1 \Rightarrow p_{\text{esc}} = 0 \Rightarrow$  the random walk is recurrent.

ii)  $m < +\infty \Rightarrow F(1) < 1 \Rightarrow p_{\text{esc}} \neq 0 \Rightarrow$  the random walk is transient

Let us then go to estimate the sum  $\sum_{n \geq 0} u_n$ .

$d=1$ : to return to the origin we need the same number of steps of type +1 than -1:

$$u_{2n+1} = 0 \quad \text{and} \quad 1^{-2n} \cdot \dots \cdot 1_1 = (2n)! 2^{-2n} \Rightarrow u_{2n} = \frac{(2n)!}{n! n!} 2^{-2n} \approx$$

$$\text{Finally, } \sum_{n \geq 1} u_n = \sum_{n \geq 1} \frac{1}{\sqrt{n} n!} = +\infty.$$

d=2: the combinatorics here is slightly more involved. We have the same number of steps ↑, ↓ and the same number ←, →. Hence:

$$\begin{aligned} u_{2n} &= \left(\frac{1}{4}\right)^{2n} \cdot \# \text{ paths with } k \uparrow, k \downarrow, r \leftarrow, r \rightarrow, \text{ such that } 2k+2r=2n. \\ &= \left(\frac{1}{4}\right)^{2n} \cdot \sum_{k=0}^n \binom{2n}{k, k, n-k, n-k} = \left(\frac{1}{4}\right)^{2n} \sum_{k=0}^n \frac{(2n)!}{k! k! (n-k)! (n-k)!} \\ &= \left(\frac{1}{4}\right)^{2n} \sum_{k=0}^n \frac{(2n)!}{n! n!} \frac{(n!)^2}{(k!)^2 (n-k!)^2} = \left(\frac{1}{4}\right)^{2n} \sum_{k=0}^n \binom{n}{k}^2 \binom{2n}{n} \end{aligned}$$

Now we can use the combinatorial identity  $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$ , so  $u_{2n} = \left(\frac{1}{4}\right)^{2n} \binom{2n}{n}^2$ , so, now by applying Stirling again we have that

$$\left(\frac{1}{4}\right)^{2n} \binom{2n}{n}^2 = \left(\frac{1}{4^n} \binom{2n}{n}\right)^2 \approx \frac{1}{\pi n} = \sum_{n \geq 1} u_n \approx \sum \frac{1}{\pi n} \rightarrow +\infty.$$

d=3: Now the expression become longer. In this case we have

$$\begin{aligned} u_{3n} &= \left(\frac{1}{6}\right)^{2n} \sum_{\substack{j \neq k \\ j+k=n}} \binom{2n}{j, j, k, k} = \left(\frac{1}{6}\right)^{2n} \sum_{\substack{j \neq k \\ j+k=n}} \frac{(2n)!}{j! j! k! k! (n-j-k)! (n-j-k)!} = \\ &= \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \sum_{\substack{j \neq k \\ j+k=n}} \left(\frac{1}{3^n} \frac{n!}{j! k! (n-k-j)!}\right)^2 = \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \sum_{\substack{j \neq k \\ j+k=n}} \left(\frac{1}{3^n} \binom{n}{j, k}\right)^2 \end{aligned}$$

Now, the step to continue is to observe that  $\binom{n}{j, k}$  is maximized when  $j, k$  and  $(n-k-j)$  are as close to  $\frac{n}{3}$  as possible. With this we get that:

$$u_{3n} = \dots \leq \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \frac{1}{3^n} \frac{n!}{\left[\frac{n}{3}\right]!^3} \sum_{\substack{j \neq k \\ j+k=n}} \frac{1}{3^n} \binom{n}{j, k}$$

Finally,  $\sum_{j \neq k} \frac{1}{3^n} \binom{n}{j, k} = 1$ ; this can be computed by developing the power  $(x+y+z)^3$ , and substituted  $j=k=x=y=z=1$ . So, we have obtained that

$$u_{3n} \leq \frac{1}{2^{2n}} \binom{2n}{n} \left(\frac{1}{3^n} \frac{n!}{\left[\frac{n}{3}\right]!^3}\right)$$

Using again Stirling, we get that:

$$\begin{aligned} \frac{1}{2^{2n}} \binom{2n}{n} \frac{1}{3^n} \frac{n!}{\left[\frac{n}{3}\right]!^3} &\approx \frac{1}{\cancel{\frac{1}{2^{2n}}}} \frac{\cancel{\frac{1}{\sqrt{2\pi n}}} \left(\frac{x}{\cancel{\sqrt{2\pi n}}} \right)^{\cancel{x}}}{\cancel{\frac{1}{\sqrt{2\pi n}}} \cancel{\left(\frac{x}{\sqrt{2\pi n}}\right)^{\cancel{x}}}} \frac{1}{\cancel{x}^2} \frac{\cancel{\frac{1}{\sqrt{2\pi n}}} \left(\frac{x}{\sqrt{2\pi n}} \right)^{\cancel{x}}}{\cancel{\left(\sqrt{2\pi \frac{n}{3}}\right)^3} \left(\frac{x}{\cancel{\sqrt{2\pi n}}}\right)^{\cancel{x}}} \\ &\approx \frac{C}{n^{\frac{3}{2}}} \Rightarrow \sum u_{3n} < +\infty \Rightarrow \text{transient.} \end{aligned}$$

Obs/less is known concerning self-avoiding walks, even the counting.