

Martingales and the convergence theorem

We finish the course with the definition of martingales. These objects are very useful because once a sequence of random variables is identified as a martingale, their convergence is guaranteed. An example will motivate the philosophy of the definition.

Ex/ A gambler with a lot of money use the following strategy: he starts wagering 1€ on an event bet. If he loses, he wagers 2€, and so on. In the case he loses consecutively N times, in the $(N+1)$ th time he bets 2^{N+1} €. Each win is calculated in such a way his ultimate win covers the loss, and profit with 1€.

Let N be the first time that is winning; then $p(N=n) = \left(\frac{1}{2}\right)^n$, and $p(N < +\infty) = 1$, which means that at some moment the gambler will win. However, the amount of money of lost L in average is equal to

$$E[L] = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n (1+2+\dots+2^{n-1}) = +\infty.$$

The philosophy from this example is the following: a gambler plays a game with capital at n -th step equal to S_n . Before the $(n+1)$ -wager, the player only knows S_0, \dots, S_n and only can guess S_{n+1}, \dots . If the game is fair, he would expect that there is no change on his present capital, namely:

$$E[S_{n+1} | S_0, S_1, \dots, S_n] = S_n.$$

This gives us the definition of martingale:

Def/ A sequence of r.v. $\{S_n\}_{n \geq 1}$ is a martingale with respect to the sequence $\{X_n\}_{n \geq 1}$ if, $\forall n \geq 1$,

a) $E[|S_n|] < +\infty$

b) $E[S_{n+1} | X_0, X_1, \dots, X_n] = S_n$

We will generally study martingales with respect to themselves, which translate into the condition $E[S_{n+1} | S_0, S_1, \dots, S_n] = S_n$. But before, let us understand better the notion of $E[*|*]$.

Def/ For two random variables X, Y , we define $E[X|Y=y]$ as

$$E[X|Y=y] = \sum_{x \in X} x P(X=x|Y=y) = f(y)$$

Then, $E[X|Y] = f(Y)$.

Similarly, $E[X|Y_0=y_0, Y_1=y_1, \dots, Y_k=y_k] = f(y_0, y_1, \dots, y_k)$ and $E[X|Y_0, Y_1, \dots, Y_k] = f(Y_0, Y_1, \dots, Y_k)$. In particular, we have the following properties:

lemma/ The following properties hold:

i) $E[U+V|Y] = E[U|Y] + E[V|Y]$

ii) If X is independent of Y , then $E[X|Y] = E[X]$

proof/ we prove first ii): observe that $E[X|Y=y] = \sum_{x \in X} x P(X=x|Y=y) = E[X]$ for all $y \Rightarrow E[X|Y] = E[X]$.

ii) $E[U+V|Y=y] \Rightarrow$ Apply linearity of the expectation.

We can now immediately obtain examples of martingals:

Ex 1/ Let X_1, X_2, \dots be independent variables with 0 means. Assume also that $\mathbb{E}[|X_n|] < \infty$. Define $S_n = X_1 + \dots + X_n$. Then:

$$\begin{aligned}\mathbb{E}[S_{n+1} | X_1, \dots, X_n] &= \mathbb{E}[S_n + X_{n+1} | X_1, \dots, X_n] = \\ &= \mathbb{E}[S_n | X_1, \dots, X_n] + \mathbb{E}[X_{n+1} | X_1, \dots, X_n] = S_n + 0 = S_n \\ &= \sum_{i=1}^{n+1} \mathbb{E}[X_i | X_1, \dots, X_n] = \sum_{i=1}^n X_i\end{aligned}$$

What is the connection between martingals and Markov chains? Well:

- ① Markov chains: the distribution at the present moment only depends on the previous state.
- ② Martingals: the mean of the future (S_{n+1}) depends only on the present state.

Ex / Take a random variable X_0 , and a sequence $\{\epsilon_n\}_{n \geq 0}$ of identically distributed random variables with $\mathbb{E}[\epsilon_n] = 0$, and independent of X_0 . Define

$$X_{n+1} = X_n + \epsilon_{n+1} X_0$$

Then we have that:

$$\begin{aligned}\bullet \mathbb{E}[X_{n+1} | X_0, \dots, X_n] &= \mathbb{E}[X_n | X_0, \dots, X_n] + \mathbb{E}[\epsilon_{n+1} X_0 | X_0, \dots, X_n] \\ &= X_n + \mathbb{E}[\epsilon_{n+1} | X_0, \dots, X_n] \mathbb{E}[X_0 | X_0, \dots, X_n] \\ &= X_n + \mathbb{E}[\epsilon_{n+1}] X_0 = X_n \quad \Rightarrow \text{It is a martingale}\end{aligned}$$

$\bullet P(X_n + \epsilon_{n+1} X_0 = a | X_n = a_n, X_{n-1} = a_{n-1}, \dots) \neq P(X_n + \epsilon_{n+1} X_0 = a | X_n = a_n)$, because X_0 could take different values.

The main interest for the martingals is that we have always hitting theorems:

Theorem / If $\{S_n\}_{n \geq 0}$ is a martingale with respect to $\{X_n\}_{n \geq 0}$ with $\mathbb{E}[S_n^2] < M < +\infty$ for some M and all n . Then there exist a random variable S such that $S_n \xrightarrow{a.s.} S$.