

L_p spaces

We now show how to dotate the set $L(X, \mathcal{X}, \mu)$ with a Banach space structure: given $f \in L(X, \mathcal{X}, \mu)$, we define $N_\mu(f)$ as

$$N_\mu(f) = \int |f| d\mu$$

Prop / $L(X, \mathcal{X}, \mu)$ is a vectorial space over \mathbb{R} , N_μ is a seminorm in $L(X, \mathcal{X}, \mu)$ and $N_\mu(f) = 0$ iff $f = 0$ μ -a.a.

Proof / We have already seen that $L(X, \mathcal{X}, \mu)$ is a vectorial space. Additionally:

$$N_\mu(\alpha f) = |\alpha| N_\mu(f), \quad N_\mu(f+g) \leq N_\mu(f) + N_\mu(g); \quad N_\mu(f) \in [0, +\infty] \quad \forall f \in L(X, \mathcal{X}, \mu)$$

Hence, in order to make $L(X, \mathcal{X}, \mu)$ to be a normed space we need to identify functions which differ in a set of measure 0. More precisely, we write $f \sim g$ for $f, g \in L(X, \mathcal{X}, \mu)$ iff $f = g$ μ -a.a.

Lemma / \sim is an equivalence relation.

Proof / Reflexive and symmetric property are obvious. Let us show transitive property. Assume that $f \sim g$ and $g \sim h$, hence $f(x) = g(x)$ if $x \notin N_1$, $\mu(N_1) = 0$, and $g(x) = h(x)$ if $x \notin N_2$, $\mu(N_2) = 0$. Hence $f(x) = h(x)$ if $x \notin N_1 \cup N_2$:

$$0 \leq \mu(N_1 \cup N_2) = \int \mathbb{I}_{N_1 \cup N_2} \leq \int \mathbb{I}_{N_1} + \mathbb{I}_{N_2} = \mu(N_1) + \mu(N_2) = 0$$

Def / $L(X, \mathcal{X}, \mu)/\sim = L_1(X, \mathcal{X}, \mu)$ and it is called Lebesgue space; if $[f]$ is a representative of the class, then

$$\|[f]\|_1 = \int |f| d\mu.$$

Prop / The previous definition does NOT depend on the representative

Proof / let f and g be 2 representatives of the class $[f]$, and so $f = g$ on $M = N^c$, $\mu(N) = 0$. As

$$\begin{aligned} \int |f| &= \int |f| \mathbb{I}_M + |f| \mathbb{I}_N = \int |f| \mathbb{I}_M + \int |f| \mathbb{I}_N = \int |g| \mathbb{I}_M + \int |g| \mathbb{I}_N \\ &= \int |g| \mathbb{I}_M + |g| \mathbb{I}_N = \int |g| \end{aligned}$$

the integral does NOT depend on the representative.

We can generalise the L_1 -space in the following way: from now on we write f instead of $[f]$:

Def / let p be a real number, $1 \leq p < +\infty$. The space $L_p = L_p(X, \mathcal{X}, \mu)$ is the set of equivalence classes \sim of the type $[f]$ such that $|f|^p$ is integrable. If $f \in L_p$, then

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}$$

Def / For a measurable set E , we write $L_p(E)$ the set of all classes of functions such that $\left(\int_E |f|^p d\mu \right)^{1/p} < +\infty$
 i.e. $\|f\|_p(E) := \left(\int_E |f|^p d\mu \right)^{1/p}$ with $\max_{x \in E} |f(x)|^p \leq 1$. In fact, $\|f\|_p \leq \|f\|_p(E)$

Proof / We use the following fact: if $A, B \geq 0$, then $AB \leq \frac{A^p}{p} + \frac{B^q}{q}$. Assume that $\|f\|_p = 0$, $\|g\|_q = 0$, otherwise $f \cdot g = 0$ and all is trivial.

The product $f \cdot g$ is X -measurable, and writing $A = \frac{\|f\|_p}{\|f\|_p \|g\|_q}$, $B = \frac{\|g\|_q}{\|f\|_p \|g\|_q}$, we have that

$$\frac{\|f \cdot g\|_1}{\|f\|_p \|g\|_q} \leq \frac{\|f\|^p}{p \|f\|_p^p} + \frac{\|g\|^q}{q \|g\|_q^q}$$

as both $\|f\|^p$ and $\|g\|^q$ are integrable with $\|f \cdot g\|_1$. Hence $f \cdot g \in L_1$. Then, taking the integral,

$$\frac{\|f \cdot g\|_1}{\|f\|_p \|g\|_q} \leq \frac{1}{p \|f\|_p^p} \int \|f\|^p + \frac{1}{q \|g\|_q^q} \int \|g\|^q = \frac{1}{p} + \frac{1}{q} = 1$$

Def / A pair of real positive numbers satisfying that $\frac{1}{p} + \frac{1}{q} = 1$ are called conjugated.

Corollary / (Cauchy-Schwarz) If $f, g \in L_2$, then $\|f \cdot g\|_1 \leq \|f\|_2 \|g\|_2$

Proof / Trivial

Lemma / (Minkowski inequality) If $f, g \in L_p$, $p \geq 1$, then $f+g \in L_p$ and $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Proof / The case $p=1$ has been already studied. Assume that $p > 1$. As f and g are measurable, also $f+g$: and then

$$\|f+g\|^p \leq (2 \max\{|f|, |g|\})^p \leq 2^p (\|f\|^p + \|g\|^p) \quad \textcircled{A}$$

and so $\|f+g\|_p \leq \infty \Rightarrow f+g \in L_p$. Now, $\|f+g\|^p = \|f+g\| \|f+g\|^{p-1} \leq \|f\| \|f+g\|^{p-1} + \|g\| \|f+g\|^{p-1}$. As $f+g \in L_p$, then $\|f+g\|^{p-1} \in L_1$. Take g conjugate of p . Then:

$$p+q=pq \quad +\infty > \int \|f+g\|^p = \int \|f+g\|^{(p-1)q} = \int \|f+g\|^{p-1}|^q = \|f+g\|^{p-1} \in L_q$$

Applying more Hölder's inequality we have (for \textcircled{A}):

$$\begin{aligned} f \in L_p \quad &\left(\Leftrightarrow \int |f| |f+g|^{p-1} \leq \|f\|_p \left(\int |f+g|^{(p-1)q} \right)^{1/q} = \|f\|_p \left(\int |f+g|^p \right)^{1/q} \right. \\ (f+g)^{p-1} \in L_q \quad &\left. = \|f\|_p \|f+g\|_p^{p/q} \right) \end{aligned}$$

Similarly happens for the term \textcircled{B} . Putting all together we get

$$\|f+g\|_p^p \leq \|f\|_p \|f+g\|_p^{p/q} + \|g\|_p \|f+g\|_p^{p/q} = \frac{1}{p} \|f\|_p + \frac{1}{q} \|g\|_p \|f+g\|_p^{p/q}$$

Finally, if $\|f+g\|_p = 0$, all is trivial; if not, then $\|f+g\|_p^{p-p/q} \leq \|f\|_p + \|g\|_p$, but again $p-p/q = 1$

Theorem / $(L_p, \|\cdot\|_p)$ is a normed space.

Theorem / (Pierz-Fischer) $(L_p, \|\cdot\|_p)$ is a Banach space.