

Now we have a fundamental theorem

Thm 1 (Carathéodory's extension Theorem) The set A^* is a σ -algebra containing A , and hence (X, A^*) is a measurable space. Additionally if $\{E_n\}_{n \geq 1}$ is a disjoint sequence in A^* , then

$$\mu^* \left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu^*(E_n)$$

Borel sets Lebesgue sets Vitali set!

In the case of \mathbb{R} we have that $\mathcal{F} \not\subseteq \mathcal{B} \not\subseteq \mathcal{F}^* \not\subseteq \mathcal{P}(\mathbb{R})$, and in particular the restriction of μ^* in \mathcal{B} gives a measure, that we call Lebesgue measure and we denote it by λ .

The Integral over positive functions

Let (X, Σ, μ) a measure space. Define $H^+(X, X) = \{f \in H(X, X) : f(x) \geq 0, \forall x \in X\}$.

Def / $\varphi : X \rightarrow \mathbb{R}$ is a simple function if $|\text{Im } \varphi| < \infty$.

In particular, if φ is measurable, $\text{Im } \varphi = \{\alpha_1, \dots, \alpha_n\}$, and $\varphi^{-1}(\alpha_i) = E_i$, then

$$\varphi(x) = \sum_{i=1}^n a_i \mathbb{I}_{E_i}(x), \quad E_i \in \mathcal{X}.$$

Obs/ A single function can be written in several ways. The previous one (where the E_i 's are disjoint) is called canonical.

We can define now the integral of a single function $\varphi \in M^+(X, \chi)$:

Def / Given $\varphi \in \mathcal{H}^+(X, X)$ simple function in its canonical form $\varphi(x) = \sum_{i=1}^n a_i \mathbb{I}_{E_i}$ and a measure μ , we define the integral of φ with respect to μ as

$$\int \varphi \, d\mu = \sum_{j=1}^n a_j \mu(E_j)$$

Obs/ ① If (X, \mathcal{X}, μ) is fixed (and hence μ it is) we write $\int \varphi d\mu = \int \varphi$

② The integral is well-defined assuming $0 \cdot (+\infty) = 0$: as $a_f > 0$, hence we do not have $+\infty + (-\infty)$.

③ for $\varphi(x) = \mathbb{1}_E(x)$ we have that $\int \varphi(x) d\mu = \mu(E)$.

Let us prove some basic properties of the integral:

Lemma / let (X, \mathcal{X}, μ) a measure space.

- a) If $c > 0$ and $\varphi \in H^+(X, X)$ is simple, then $\int c\varphi = c \int \varphi$
 b) If $\varphi, \psi \in H^+(X, X)$ are simple, then $\int \varphi + \psi = \int \varphi + \int \psi$
 c) If $\varphi \in H^+(X, X)$ is simple, then $\int \varphi \, I_{E \in \Delta} \mu = P(E)$ is a measure over X ($E \in \mathcal{X}$)

Proof / We prove a) and c).

a) If $c=0$, then $c\varphi=0$ and $0 \cdot \int \varphi = 0 \cdot * = 0$. If $c > 0$, then $c\varphi \in \mathcal{M}^+(X, X)$ with canonical form $c\varphi = c \sum_{i=1}^n a_i \mathbb{I}_{E_i} = \sum_{i=1}^n c a_i \mathbb{I}_{E_i} = 0$

$$\int c\varphi = \sum_{i=1}^n (c a_i) \mu(E_i) = c \sum_{i=1}^n a_i \mathbb{I}_{E_i} = c \int \varphi$$

c) We write $\varphi \mathbb{I}_E = \sum_{i=1}^n a_i \mathbb{I}_{E_i} \mathbb{I}_E = \sum_{i=1}^n a_i \mathbb{I}_{E_i \cap E} + 0 \cdot \mathbb{I}_{E^c}$. Using the linearity we have that

$$\rho(E) = \int \varphi \mathbb{I}_E d\mu = \int \sum_{i=1}^n a_i \mathbb{I}_{E_i \cap E} d\mu = \sum_{i=1}^n \mu(E_i \cap E) = \sum_{i=1}^n \mu_i(E),$$

where $\mu_i(E) = \mu(E \cap E_i)$. This is clearly a measure. In the problem sheet 1 we show that the sum of measures is also a measure, so the result holds.

We can now define the integral in general terms: let (X, \mathcal{X}, μ) be a measure space, $f \in \mathcal{M}^+(X, X)$. Then we write

$$\int f d\mu = \sup_{\varphi} \int \varphi d\mu \in [0, +\infty] \cup \{-\infty\}$$

where \sup is taken over all simple functions $\varphi \in \mathcal{M}^+(X, X)$ such that $0 \leq \varphi(x) \leq f(x)$, $\forall x \in X$.

Def/ Let $E \in \mathcal{X}$, $f \in \mathcal{M}^+(X, X)$. We define the integral of f over E as

$$\int_E f d\mu = \int \underbrace{f \mathbb{I}_E}_{\in \mathcal{M}^+(X, X)} d\mu$$

Lemma/ Let $f, g \in \mathcal{M}^+(X, X)$, $E, F \in \mathcal{X}$.

i) If $f \leq g$, then $\int f \leq \int g$

ii) If $E \subset F$, then $\int_E f \leq \int_F f$

Proof/ i) If $\varphi \in \mathcal{M}^+(X, X)$ is a simple function such that $0 \leq \varphi \leq f$, then, $0 \leq \varphi \leq g$. Hence,

$$\int f = \sup_{\varphi \leq f} \int \varphi = \sup_{\varphi \leq g} \int \varphi = \int g.$$

ii) As $f \mathbb{I}_E \leq f \mathbb{I}_F$, we can apply the previous result.

Now we can state and prove the most important result in $\mathcal{M}^+(X, X)$:

Theorem/ (Monotone Convergence Theorem) Let (X, \mathcal{X}, μ) be a measure space and $\{f_n\}_{n \geq 1}$ be a sequence of functions in $\mathcal{M}^+(X, X)$ monotone increasing (i.e., $\forall n, \forall x \in X$, $f_n(x) \leq f_{n+1}(x)$). If $\{f_n\}_{n \geq 1} \rightarrow f$ pointwise, then

$$\int f = \lim \int f_n$$

Proof/ We proceed that if $\{f_n\}_{n \geq 1} \subseteq \mathcal{M}(X, X)$, $\{f_n\}_{n \geq 1} \rightarrow f$ pointwise, then $f \in \mathcal{M}(X, X)$ and hence $f \in \mathcal{M}^+(X, X)$. So $\int f$ has sense. As $f_n \leq f$ $\forall n$ we have that $\forall n \int f_n \leq \int f$, and so

$$\lim \int f_n \leq \int f$$

Let us prove now the " \geq " statement. Take $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, and let $\varphi \in \mathcal{M}^+(X, X)$ simple

such that $0 \leq \varphi \leq f$. Define $A_n = \{x \in X : f_n(x) \geq \alpha \varphi(x)\}$. It is clear that:

- $f_n(x) - \alpha \varphi(x)$ is X -measurable, hence $A_n = \{x \in X : f_n(x) - \alpha \varphi(x) \in [0, +\infty)\} \subset X$.
- $A_n \subseteq A_{n+1}$ because $f_{n+1}(x) \geq f_n(x) \forall n, \forall x \in X$.
- $X = \bigcup A_n$: if there is $x \in X$, such that $x \notin A_n \forall n$, then $\forall n f_n(x) < \alpha \varphi(x)$; but $n \geq 1 \Rightarrow \alpha \varphi(x) \leq \alpha f(x)$, and hence

$$\forall n \quad f_n(x) < \alpha f(x) \quad (0 < \alpha < 1) \Rightarrow \lim_n f_n(x) \neq f(x)$$

Now we have the following:

$$\alpha \int_{A_m} \varphi = \alpha \int \varphi \mathbb{I}_{A_m} = \int \alpha \varphi \mathbb{I}_{A_m} = \int_{A_m} \alpha \varphi \stackrel{\uparrow}{\leq} \int_{A_n} f_n \stackrel{\downarrow}{\leq} \int_{A_n} f_n$$

On the other side we have that

$$\begin{aligned} \int \varphi &= \int_{\bigcup_{m \geq 1} A_m} \varphi = \int \varphi \mathbb{I}_{\bigcup_{m \geq 1} A_m} = \rho \left(\bigcup_{m \geq 1} A_m \right) \stackrel{\downarrow}{=} \lim_n \rho(A_n) = \\ &= \lim_n \int \varphi \mathbb{I}_{A_m} = \lim_n \int_{A_n} \varphi \end{aligned}$$

so, we have shown that

$$\alpha \int \varphi = \lim_n \alpha \int_{A_n} \varphi \leq \lim_n \int_{A_n} f_n \stackrel{\text{we take}}{\underset{\text{or adding}}{\Rightarrow}} \int \varphi \leq \lim_n \int_{A_n} f_n$$

And finally, this is true for all φ , hence $\int f = \sup_{\varphi \leq f} \int \varphi \leq L \int f$. ■

Let us see some consequences of the (MCT):

Corollary / Let $c > 0$, $f, g \in \mathcal{M}^+(X, \mathbb{X})$. Then $\int c f = c \int f$ and $\int (f+g) = \int f + \int g$.

Proof / For $c = 0$ it is obvious. If $c > 0$, let $\{f_n\}_{n \geq 1}$ be a monotone increasing sequence which converge to f (we have built such a function). Hence,

$$\int c f = \lim_n \int c f_n = c \lim_n \int f_n = c \int f.$$

We proceed similarly for f, g , by taking sequences $\{f_n\}_{n \geq 1}, \{g_n\}_{n \geq 1}$ monotone increasing and tending to f, g , respectively.

However, something different happen if we avoid the monotone condition:

Lemma / (Fatou's lemma) Let $\{f_n\}_{n \geq 1}$ a sequence in $\mathcal{M}^+(X, \mathbb{X})$; then $\int \liminf \{f_n\} \leq \liminf \int f_n$.

Proof / Take $g_m(x) = \inf \{f_m(x), f_{m+1}(x), \dots\}$ for all $x \in X$. In this way, if $m \leq n$, $g_m \leq f_n$. We know that $g_m \in \mathcal{M}^+(X, \mathbb{Y})$, and so

$$\forall n \geq m \Rightarrow \int g_m \leq \int f_n \Rightarrow \int g_m \leq \liminf \int f_n$$

The sequence $\{g_m\}_{m \geq 1}$ is monotone increasing with limit $\liminf \{f_n\}$. Hence, by the MCT we have that

$$\lim \int g_m = \int \lim g_m = \int \liminf \{f_n\} \Rightarrow \int \liminf \{f_n\} \leq \liminf \int f_n \quad \text{■ (5)}$$

Corollary / If $f \in M^+(X, \mathcal{X})$, then $\rho(f) = \int_E f$ is a measure over X .

Proof / Proving that $\rho(f) \geq 0 \ \forall E \in \mathcal{X}$ and $\rho(\phi) = 0$ is trivial. Take now $\{E_n\}_{n=1}^{\infty}$, a sequence of disjoint sets, $\bigcup_{n=1}^{\infty} E_n = E$. Write

$$f_m = \sum_{k=1}^m f \mathbb{I}_{E_k} \Rightarrow \int f_m = \sum_{k=1}^m \int f \mathbb{I}_{E_k} = \sum_{k=1}^m \rho(E_k)$$

As $\{f_n\}_{n=1}^{\infty} \rightarrow f \mathbb{I}_E$ in a monotone increasing way, we have that

$$\rho\left(\bigcup_{n=1}^{\infty} E_n\right) = \rho(E) = \int f \mathbb{I}_E = \int \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int f_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n \rho(E_k) = \sum_{n=1}^{\infty} \rho(E_n)$$

Corollary / Let (X, \mathcal{X}, μ) be a measurable space and $f \in M^+(X, \mathcal{X})$. Then

$$\int f d\mu = 0 \Leftrightarrow f = 0 \text{ } \mu\text{-almost always}$$

Proof / \Rightarrow Assume that $\int f = 0$, and define $E_n = \{x \in X : f(x) \geq \frac{1}{n}\}$. In particular, the sequence $\{E_n\}_{n=1}^{\infty}$ is increasing and $f \geq \frac{1}{n} \mathbb{I}_{E_n}$. Then

$$0 = \int f \geq \int \frac{1}{n} \mathbb{I}_{E_n} = \frac{1}{n} \mu(E_n) \geq 0 \Rightarrow \mu(E_n) = 0 \ \forall n.$$

Now, $E = \{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$, and so $\mu(E) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \stackrel{\text{increasing}}{=} \lim \mu(E_n) = \lim 0 = 0$

\Leftarrow Assume that $f = 0$ μ -a.a., and consider again $E = \{x \in X : f(x) > 0\}$. So $\mu(E) = 0$. Define $f_n = n \mathbb{I}_E$. Then:

$$\lim f_n(x) = \begin{cases} +\infty, & f(x) > 0 \\ 0, & f(x) = 0 \end{cases} \Rightarrow f \leq \lim f_n = \liminf f_n.$$

$$\text{Now, } 0 \leq \int f \leq \int \liminf f_n \stackrel{\text{(Fatou)}}{\leq} \liminf \int f_n = \liminf \int n \mathbb{I}_E = \liminf \overbrace{n \mu(E)}^0 = 0$$

We use now this corollary to prove a straight of the MCT:

Corollary / (Extension of MCT) Let $\{f_n\}$ be a sequence in $M^+(X, \mathcal{X})$, monotone increasing towards f except in a set of measure 0. Then

$$\int f = \lim \int f_n$$

Proof / Write $N \in \mathcal{X}$, $\mu(N) = 0$ the set where $\{f_n\}$ NOT monotone increasing. Write $M = N^c$. Then $\{f_n \mathbb{I}_M\}$ is monotone increasing and tends to $f \mathbb{I}_M$. Hence,

$$\int f \mathbb{I}_N = \lim \int f_n \mathbb{I}_N$$

As $\mu(N) = 0$, the function $f_n \mathbb{I}_N$ and $f \mathbb{I}_N$ are = 0 μ -a.a., so $\int f_n \mathbb{I}_N = \int f \mathbb{I}_N = 0$. So now:

$$\int f = \int f (\mathbb{I}_M + \mathbb{I}_N) = \int f \mathbb{I}_M + \int f \mathbb{I}_N = \lim \int f_n \mathbb{I}_M + \lim \int f_n \mathbb{I}_N = \lim \int f_n$$

Corollary / Let $\{g_n\}_{n=1}^{\infty}$ be a sequence in $M^+(X, \mathcal{X})$. Then $\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n$

Proof / Just consider $f_n = \sum_{k=1}^n g_k$: it defines a monotone increasing family of functions, and we can easily ... $\forall k \in \mathbb{N}$. Now