Correction of the errata in the Kolmogorov 0-1 Law

In a past lecture we 'proved' the following Theorem, which was the cornerstone to prove later Kolmogorov's 0-1 law:

> `'Let` $\{\mathcal{A}_i\}_{i\in I}$ `be a sequence of families of events. If the sequence is independent, then the sequence` $\{\sigma(\mathcal{A}_i)\}_{i\in I}$ `is also independent'`

In the proof of this statement we used the following (wrong) argument: whenever A is independent with each element in $\{B_i\}_{i\geq 1}$, then A is independent with its union. This statement is not true and is very easy to get counterexamples.

In this note we will manage to correct this difficulty in the proof of the previous stated theorem. We will proceed in the following way:

1. Statement of the main result we need (called Dynkin's Monotone Class Theorem)

2. How we use it in our application.

3. Proof of Dynkin's theorem.

Let $\mathcal{F}$ be a family of subsets of Ω, and consider an operation $\bullet$ defined over (sets) of sets. Here we have some examples:

1. $\bullet = \cap$, this operation is a binary operation.

2. $\bullet = -$, the monotone difference between pair of sets $A \subset B$: $B - A$ is the set of elements contained in B but not in A.

3. $\bullet = \lim$, the monotone limit defined over increasing (or decreasing) families of sets.

Given an operation $\bullet$, we can define $\mathcal{F}^\bullet$ as the minimal extension of $\mathcal{F}$ (namely, $\mathcal{F} \subseteq \mathcal{F}^\bullet$) which is closed by the operation $\bullet$. Clearly, $\mathcal{F}^\bullet \subseteq 2^\Omega$, so this definition has sense.

The main result we have is the following:

Theorem 1 *(Dynkin Monotone Class Theorem) If $\Omega \in \mathcal{F}$ and is closed under $\cap$, then*

1. *The algebra generated by $\mathcal{F}$ is $\mathcal{F}^-$,*

2. *If $\mathcal{F}$ is an algebra, then $\sigma(\mathcal{F}) = \mathcal{F}^{\lim}$.*

We will see the ideas behind the proof later (are elementary), and now we see how to apply this result to our setting. First step is to state the right result we will prove:

Theorem 2 *Let $\{\mathcal{A}_i\}_{i\in I}$ be a sequence of families of events* closed under $\cap$. *If the sequence is independent, then the sequence $\{\sigma(\mathcal{A}_i)\}_{i\in I}$ is also independent.*

Before proving this result, observe that the extra condition we require ('closed under $\cap$') is also satisfied in the proof of Kolmogorov (observe that we deal with sequences of random variables, and such families are closed under intersection).

Proof: The proof follows the same ideas as we did in the lecture. The key point is based on studying the set $S(B) = \{A \in \sigma(\mathcal{A}_1) : A \,\text{is independent with}\, B\}$. Before going it, define $a(\mathcal{F})$ the algebra generated by $\mathcal{F}$ (namely, the smallest algebra which contains $\mathcal{F}$), and define (for an event B intersection of other events)

$$s(B) = \{A \in a(\mathcal{A}_1) : p(A \cap B) = p(A)p(B)\}.$$

Observe now that $s(B)$ is closed by the minus operation: if A_1, $A_2 \in s(B)$, $A_1 \subset A_2$ then

$$p((A_1 - A_2) \cap B) = P(A_1 \cap B) - p(A_2 \cap B) = p(A_1 - A_2)p(B),$$

and so $s(B)$ is closed under the operator $-$. So by Dynkin's Theorem $s(B) = s(B)^- \supset \mathcal{A}_1^- = a(\mathcal{A}_1)$. As $s(B) \subset \mathcal{A}_1$, necessarily $s(B) = a(\mathcal{A}_1)$.

Now let us show something similar, but for σ-algebras. Define now,

$$S(B) = \{A \in \sigma(\mathcal{A}_1) : p(A \cap B) = p(A)p(B)\}.$$

where again B is obtained as the intersection of events in the others $\mathcal{A}_i$. Let us see that $S(B) = \sigma(\mathcal{A}_i)$, which would prove the statement. By the first argument over algebras, $a(\mathcal{A}_1) \subset S(B)$. Now, let us show that $a(\mathcal{A}_1)$ is closed under monotone limits (and then we would be able to apply the second part of Dynkin's Theorem), which will give that $a(\mathcal{A}_1)^{\lim} = \sigma(\mathcal{A}_1)$. This is true because $a(\mathcal{A}_1) \subset S(B) \subset \sigma(\mathcal{A}_1)$.

Let $\{A_i\}_{i\geq 1}$ be a monotone (either increasing or decreasing) sequence of events from $a(\mathcal{A}_1)$ with limit A. Let $A^{(k)}$ the k-th union or the k-th intersection depending on the type of monotonicity. Then, due to the properties of $s(B)$ we have that:

$$p(A \cap B) = \lim p(A^{(k)} \cap B) = \lim p(A^{(k)}) \cap p(B) = p(A)p(B),$$

hence, $\sigma(\mathcal{A}_1) = a(\mathcal{A}_1)^{\lim} \subset S(B)^{\lim} = S(B)$ and this means that $S(B) = S(B)^{\lim} = \sigma(\mathcal{A}_1)$, as we wanted to show.

□

The proof of the Monotone Class Theorem

Let us see the main ideas of the proof.

Proof of 1. Assume that $\mathcal{F}$ contains Ω and it is closed under intersection). As the algebra generated by $\mathcal{F}$ is closed under intersection, it is just needed to show that $\mathcal{F}^-$ is an algebra.

It is obvious that $\Omega \in \mathcal{F}^-$, because $\Omega \in \mathcal{F}$ and the empty set is also in $\mathcal{F}^-$ because is the difference of any set with itself. Also, if $A \in \mathcal{F}$, then $\overline{A} = \Omega - A$, and so $\overline{A} \in \mathcal{F}^-$.

We need to finish showing that $A, B \in \mathcal{F}^-$ then $A \cup B \in \mathcal{F}^-$, which extends to any finite union. This is equivalent to prove that $A, B \in \mathcal{F}^-$ then $A \cap B \in \mathcal{F}^-$, because $A \cup B = \overline{\overline{A} \cap \overline{B}}$. Assume first that we fix $A \in \mathcal{F}$ and we define

$$S_A = \{B \subseteq \Omega : A \cap B \in \mathcal{F}^-\}.$$

Is clear that $\mathcal{F} \subseteq S$ because $\mathcal{F}$ is closed under intersection. Let us see that S_A is closed under monotone differences: if B_1, B_2 are sets in S_A, $B_1 \subset B_2$, then

$$A \cap (B_2 - B_1) = (A \cap B_2) - (A \cap B_1) \in \mathcal{F}^-$$

and so $B_2 - B_1 \in S_A$. In other words, $\mathcal{F}, \mathcal{F}^- \subseteq S_A$. Finally, observe that similar arguments work when choosing $A \in \mathcal{F}^-$ instead of $\mathcal{F}$. This shows that $\mathcal{F}^-$ is closed under intersection as we wanted to prove.

Proof of 2. We do not do the details. The idea is to show first that $\mathcal{F}^{\lim}$ is a σ-algebra. Once this is done, one uses a similar argument as in 1. to show that $\mathcal{F}^{\lim}$ is closed under intersection.