

Integral for general functions. The Dominated convergence Theorem.

Next step is to study the integral of functions with both positive and negative values. In such case, it is more convenient to study functions which take values in \mathbb{R} (instead of \mathbb{R}^k).

Def 1 The set $L(X, X, \mu)$ of integrable functions is given by functions $f: X \rightarrow \mathbb{R}$ which are X -measurable and such that f^+ and f^- have finite integral with respect to μ .

Def 1 Let $f \in L(X, X, \mu)$. We define the integral of f with respect to μ as

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

All the basic properties for this integral are deduced from the properties deduced for integral of positive functions.

Prop 1 f measurable, $f \in L(X, X, \mu)$ iff $|f| \in L(X, X, \mu)$. Then $|\int f| \leq \int |f|$.

Proof / \Rightarrow If $f \in L(X, X, \mu)$, then f^+ and $f^- \in M(X, X, \mu)$ and their integral is finite. As $|f|^+ = f^+$ and $|f|^- = 0$, and $|f| = f^+ + f^-$, then both $|f|^+$ and $|f|^- \in M(X, X, \mu)$ with finite integral. Hence, if $f \in L(X, X, \mu) \Rightarrow |f| \in L(X, X, \mu)$.

\Leftarrow As $f^+ \leq |f|^+$ and $f^- \leq |f|^+$, then if $|f| \in L(X, X, \mu)$ then $f \in L(X, X, \mu)$.

Finally,

$$\begin{aligned} |\int f| &= \left| \int f^+ - \int f^- \right| \leq \left| \int f^+ \right| + \left| \int f^- \right| = \int f^+ + \int f^- = \int |f|^+ = \\ &= \int |f|^+ - \int |f|^- = \int |f| \end{aligned}$$

Corollary 1 let f be a measurable function in (X, X, μ) , $g \in L(X, X, \mu)$ and $|f| \leq g$. Then $f \in L(X, X, \mu)$ and

$$\int |f| \leq \int |g|$$

Proof If f is measurable, then $|f|$ is also measurable, and as $|f| \leq g$ we know that $|f| \leq \int |g|$. As the second term is finite, the first it is also, hence $|f| \in L(X, X, \mu)$ and $f \in L(X, X, \mu)$.

Obs 1 If $f, g \in L(X, X, \mu)$, $a \in \mathbb{R}$ then $a f + g \in L(X, X, \mu)$ and $\int a f + g = a \int f + \int g$

Theorem / (Dominated Convergence Theorem) Let $\{f_n\}_{n \in \mathbb{N}}$ a sequence of measurable functions which converge to f μ -a.a. If it exists $g \in L(X, X, \mu)$ such that $|f_n| \leq g$ $\forall n$ then $f \in L(X, X, \mu)$ and

$$\int f = \lim_{n \rightarrow \infty} \int f_n$$

Proof Redefining f and all f_n in the set $E, \mu(E) = 0$ where the limit differs, we can assume that the convergence $f_n \rightarrow f$ is true in all X , and that does not change the value of the integrals. As f_n are measurable and $|f_n| \leq g$, then f_n are integrable. Indeed, as f is the limit of measurable functions, then it is measurable. Also, as $|f_n| \leq g$, then $|f| \leq g$, and consequently f is also integrable.

As $g + f_n \geq 0$, we have the following:

$$\begin{aligned} \int g + f = \int g + f_n &= \int g + \liminf f_n = \int \liminf f_n \stackrel{\text{Fatou}}{\leq} \\ &\leq \liminf \int g + f_n = \liminf \int g + \liminf \int f_n = \int g + \liminf \int f_n. \end{aligned}$$

And now, as $\int g < \infty$, $\int f \leq \liminf \int f_n$. Now we apply a similar argument with $g - f_n \geq 0$, getting that:

$$\int g - f = \int g - f_n \stackrel{\text{Fatou}}{\leq} \liminf \int (g - f_n) = \int g - \limsup \int f_n \stackrel{\text{Fatou}}{\leq} \limsup \int f_n \leq \int f$$

$$\text{And so } \limsup \int f_n \leq \int f \leq \liminf \int f \Rightarrow \int f = \limsup \int f_n = \liminf \int f_n = \int f_n.$$

Integrals which depend on parameters

We can use the DCT in order to prove a wide variety of results. In the following, $f : X \times [a,b] \rightarrow \mathbb{R}$, and we will assume that for each choice of $t_0 \in [a,b]$, $f_{t_0} : X \xrightarrow{\text{measurable}} \mathbb{R}$, $f_{t_0}(x) = f(x, t_0)$ is X measurable.

All the time, the integrals are with respect to x .

Lemma / Assume that for some value $t_0 \in [a,b]$, $f(x, t_0) = \lim_{t \rightarrow t_0} f(x, t)$ $\forall x \in X$, and such that $\exists g \in L(X, X, \mu)$ with $|f(x, t)| \leq g(x)$ $\forall t \in [a,b]$, $\forall x \in X$. Then

$$\int f(x, t_0) d\mu(x) = \lim_{t \rightarrow t_0} \int f(x, t) d\mu(x)$$

Proof / Take t_n sequence in $[a,b]$, such that $t_n \rightarrow t_0$. Define $f_n(x) = f(x, t_n)$. By hypothesis we have

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} f(x, t_n) \stackrel{\text{DCT}}{=} f(x, \lim_{n \rightarrow \infty} t_n) = f(x, t_0)$$

$$\text{then, } \int f(x, t_0) d\mu(x) = \int \lim_{n \rightarrow \infty} f_n(x) d\mu(x) \stackrel{\text{DCT}}{=} \lim_{n \rightarrow \infty} \int f_n(x) d\mu(x) = \lim_{n \rightarrow \infty} \int f(x, t_n) d\mu(x)$$

$$\lim_{t \rightarrow t_0} \int f(x, t) d\mu(x).$$

Corollary / Assume that the function $t \mapsto f(x, t)$ is continuous in $[a,b]$ $\forall x \in X$. If there exist $g \in L(X, X, \mu)$ such that $|f(x, t)| \leq g(x)$ $\forall t \in [a,b]$, $\forall x \in X$ then the function

$$F(t) = \int f(x, t) d\mu(x)$$

is continuous in $[a,b]$.

Proof / Just apply the previous lemma to each point $t_0 \in [a,b]$. This gives that the limit always exist, and hence the function is continuous.

Corollary / Assume that the function $x \mapsto f(x, t_0) \in L(X, X, \mu)$ for some $t_0 \in [a,b]$, that $\frac{\partial f}{\partial t}(x, t)$ exist in $X \times (a,b)$, and that there is $g \in L(X, X, \mu)$ such that

$$\left| \frac{\partial f}{\partial t}(x, t) \right| \leq g(x), \forall t \in (a, b), \forall x \in X \Rightarrow \text{Then } F(t) = \int f(x, t) d\mu(x) \text{ satisfies}$$

$$F'(t) = \int \frac{\partial f}{\partial t}(x, t) d\mu(x) \quad \forall t \in (a, b)$$

Proof / Let $t \in (a, b)$, and take a sequence $t_n \in [a, b]$, $t_n \rightarrow t$, $t_n \neq t$. Then

$$\frac{df}{dt}(x, t) = \lim_{t_n \rightarrow t} \frac{f(x, t_n) - f(x, t)}{t_n - t}, \quad x \in X$$

The function $x \mapsto \frac{df}{dt}(x, t)$ is measurable for each t because it is the limit of measurable functions. For each choice of t we can apply the mean value theorem over the reals: $\exists s \in (t, t_0)$ such that

$$f(x, t) - f(x, t_0) = (t - t_0) \frac{df}{dt}(x, s)$$

Hence, $|f(x, t)| \leq |f(x, t_0)| + |t - t_0| g(x)$, and so $f(x, t) \in L(X, \mathbb{R}, \mu)$ for every choice of $t \in (a, b)$. Hence,

$$\frac{F(t_n) - F(t)}{t_n - t} = \int \frac{f(x, t_n) - f(x, t)}{t_n - t} d\mu(x) \quad \left| \frac{f(x, t_n) - f(x, t)}{t_n - t} \right| \leq \frac{|f(x, t_n)| + |f(x, t)|}{|t_n - t|} \leq g(x)$$

The term inside the integral is dominated by $|g| \in L(X, \mathbb{R}, \mu)$, hence,

$$\begin{aligned} \int \frac{df}{dt} d\mu(x) &= \int \lim_{n \rightarrow \infty} \frac{f(x, t_n) - f(x, t)}{t_n - t} d\mu(x) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \int \frac{f(x, t_n) - f(x, t)}{t_n - t} d\mu(x) \\ &= \lim_{n \rightarrow \infty} \frac{F(t_n) - F(t)}{t_n - t} = F'(t). \end{aligned}$$

The relation between the Riemann integral and the Lebesgue integral.

We have already built a measure (the Lebesgue measure) which generalises the length of an interval. The next question is to know the relation between the Riemann integral and the Lebesgue integral.

Theorem / If f is Riemann integrable in $[a, b]$, then $f \cdot \mathbb{I}_{[a, b]} \in L(\mathbb{R}, \mathcal{B}, \lambda)$ and

$$\int f \mathbb{I}_{[a, b]} d\lambda = \int_a^b f(x) dx$$

Proof / omitted.

Remark / If f is Riemann integrable in a bounded interval, then $|f|$ is also. This is not true in unbounded regions:

$$\int_0^{+\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \quad \text{but} \quad \int_0^{+\infty} \left| \frac{\sin x}{x} \right| dx = +\infty$$

This is something that does NOT happen in the Lebesgue integral: remember that $f \in L(X, \mathbb{R}, \mu)$ iff $|f| \in L(X, \mathbb{R}, \mu)$.

Remark / If $\int_{-\infty}^{\infty} |f(x)| dx < \infty$, then $f \in L(X, \mathbb{R}, \mu)$. As we have that $\lim_{n \rightarrow \infty} \int_{-n}^n |f(x)| dx < \infty$, we define $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} |f(x)| \mathbb{I}_{[-n, n]} dx$, and so

$$\int g_n d\lambda = \int |f| \mathbb{I}_{[-n, n]} d\lambda = \int_{-n}^n |f(x)| dx \quad (g_{n+1} \geq g_n \geq 0)$$

By applying the MCT, we have now that $+\infty > \int_{-\infty}^{\infty} |f(x)| dx = \lim_{n \rightarrow \infty} \int_{-n}^n |f(x)| dx =$

Extra notations

① The Radon-Nikodym Theorem

We already have seen the following: if $f \in H^+(X, \mathcal{X})$ then $\rho(E) = \int_E f d\mu$ a measure over X . The natural question is the following: given a measure space (X, \mathcal{X}, μ) , and another measure ρ over (X, \mathcal{X}) , can we define f as $\rho(E) = \int_E f d\mu$ for some f ?

Def/ let (X, \mathcal{X}, μ) be a measure space. Then another measure ρ is absolutely continuous w.r.t. μ iff $\mu(E) = 0$ implies that $\rho(E) = 0$.

Def/ let (X, \mathcal{X}, μ) a measure space. We say that μ is σ -finite if there is sequence $\{E_n\}_{n \in \mathbb{N}}$ such that $X = \bigcup_{n=1}^{\infty} E_n$ and $\mu(E_n) < \infty \forall n$.

We have then the following:

Theorem/ (Radon-Nikodym). let μ and ρ be two measures over (X, \mathcal{X}) . If ρ is absolutely continuous with respect to μ , μ is also σ -finite then $\exists f \in H^+(X, \mathcal{X})$ such that

$$\rho(E) = \int_E f d\mu \quad \forall E \in \mathcal{X}.$$

Furthermore, if $\rho(E) = \int_E g d\mu$ $\forall E \in \mathcal{X}$, then $f = g$ μ -a.a.

Def/ f is called the Radon-Nikodym derivative of ρ with respect to μ : $\frac{d\rho}{d\mu} = f$.

② Product of measure spaces

Given (X, \mathcal{X}, μ) and (Y, \mathcal{Y}, ν) two measure spaces, we want to build a new measure space by combining in some way both constructions. We do the following construction:

Ⓐ We take $Z = X \times Y$.

Ⓑ we take the σ -algebra generated by the elements of the form $A \times B$, where $A \in \mathcal{X}$, $B \in \mathcal{Y}$. We write this σ -algebra $X \otimes Y$, and it is the smallest one that makes the projection, $\pi_1: X \times Y \rightarrow X$ and $\pi_2: X \times Y \rightarrow Y$ measurable.

Under the assumption that μ and ν are σ -finite, we have the following result:

Theorem/ let (X, \mathcal{X}, μ) and (Y, \mathcal{Y}, ν) to measure spaces, and μ, ν are σ -finite measures. Then there exists a unique measure (that we call product measure) over $(X \times Y, X \otimes Y)$, that we write $\mu \otimes \nu$, such that

$$\forall A \in \mathcal{X}, \forall B \in \mathcal{Y}, \mu \otimes \nu(A \times B) = \mu(A)\nu(B)$$

Additionally if $C \in X \otimes Y$, $x \in X$, $C_x = \{y \in Y : (x, y) \in C\}$, $C^y = \{x \in X : (x, y) \in C\}$, then

$$\mu \otimes \nu(C) = \int_X \nu(C_x) d\mu = \int_Y \mu(C^y) d\nu$$

In particular this last theorem can be used to prove Fubini's Theorem in measure theory:

Theorem/ (Fubini's theorem) let (X, \mathcal{X}, μ) and (Y, \mathcal{Y}, ν) 2 measurable spaces, and let $f: X \times Y \rightarrow [0, +\infty]$ be a function in $L^1(X \times Y, \mu \otimes \nu)$. Then

$$\int_{X \times Y} f d(\mu \otimes \nu) = \int_X \left(\int_Y f d\nu \right) d\mu = \int_Y \left(\int_X f d\mu \right) d\nu$$