

## Boole-Cantelli Lemma and 0-1 laws

We will now state and prove a lemma with multiple applications.

**Lemma / (Boole-Cantelli)** Let  $(\Omega, \mathcal{A}, p)$  be a probability space,  $\{\mathbf{A}_n\}_{n \in \mathbb{N}}$  a sequence of events and  $A = \bigcap_{n \in \mathbb{N}} \mathbf{A}_n$ . Then

a)  $p(A) = 0$  if  $\sum_n p(\mathbf{A}_n) < +\infty$

b)  $p(A) = 1$  if  $\sum_n p(\mathbf{A}_n) = +\infty$  and  $\mathbf{A}_1, \mathbf{A}_2, \dots$  are independent events

**Proof /**

a) For any  $n$ ,  $A \subseteq \bigcup_{m=n}^{\infty} \mathbf{A}_m$ , and so  $p(A) \leq \sum_{m=n}^{\infty} p(\mathbf{A}_m) \xrightarrow{n \rightarrow \infty} 0$ . As  $n$  can be taken arbitrarily big, necessarily  $p(A) = 0$ .

b) Observe that  $\bar{A} = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \bar{\mathbf{A}}_m$ . Additionally,

$$p\left(\bigcap_{m=1}^{\infty} \bar{\mathbf{A}}_m\right) \stackrel{\text{def}}{=} \prod_{r=1}^{\infty} p\left(\bigcap_{m=r}^{\infty} \bar{\mathbf{A}}_m\right) = \prod_{r=1}^{\infty} \prod_{m=r}^{\infty} p(\bar{\mathbf{A}}_m) = \prod_{r=1}^{\infty} \prod_{m=r}^{\infty} (1 - p(\mathbf{A}_m))$$

$$\stackrel{1-x \leq e^{-x} \text{ if } x \geq 0}{\leq} \prod_{r=1}^{\infty} \prod_{m=r}^{\infty} e^{-p(\mathbf{A}_m)} = \prod_{m=1}^{\infty} e^{-p(\mathbf{A}_m)} = e^{-\sum_{m=1}^{\infty} p(\mathbf{A}_m)},$$

and this last term is equal to 0 whenever  $\sum_{m=1}^{\infty} p(\mathbf{A}_m) = +\infty$ . So  $p(\bar{A}) = \prod_{m=1}^{\infty} p\left(\bigcap_{n=m}^{\infty} \bar{\mathbf{A}}_m\right) = 0$ , getting that  $p(A) = 1$ .

Ob / b) is false if we do not take  $\{\mathbf{A}_n\}_{n \in \mathbb{N}}$  independent: choosing  $\mathbf{A}_n = E$ ,  $0 < p(E) < 1$ , we have that  $A = E$ ,  $\sum_n p(E) = +\infty$ , but  $p(A) = p(E) \neq 1$ .

If one remembers what  $A$  is ("infinitely many of the  $\mathbf{A}_n$ 's occur"), the previous lemma reads in the following way: If  $\mathbf{A}_1, \mathbf{A}_2, \dots$  are independent, then  $p(A) \in \{0, 1\}$  depending only on the sum  $\sum_n p(\mathbf{A}_n)$ . This is an example of the so-called 0-1-law. We show another example of this philosophy. We need to state first some definitions:

**Def /** Let  $(\Omega, \mathcal{A}, p)$  be a probability space, and let  $\{\mathbf{A}_i\}_{i \in I}$  be a sequence of families of events. We say that the sequence is independent if for all choice  $A_i \in \mathbf{A}_i$ ,  $\{\mathbf{A}_i\}_{i \in I}$  is independent.

**Def /**  $\{\mathbf{A}_i\}_{i \in I}$  and  $\{\mathbf{B}_j\}_{j \in J}$  are independent if  $\forall A_i \in \mathbf{A}_i, B_j \in \mathbf{B}_j$ ,  $A$  and  $B$  are independent.

This generalizes the notion of independence of events. Let us prove more this important result:

**Theorem /** Let  $\{\mathbf{A}_i\}_{i \in I}$  be sequence of families of events. If the sequence is independent, then the sequence  $\{\sigma(\mathbf{A}_i)\}_{i \in I}$  is also independent.

**Proof /** Without loss of generality we may assume that  $\Omega \in \mathbf{A}_i \forall i \in I$ . Observe that in order to check independence, we can restrict to finite sequences:  $I = \{1, \dots, n\}$ . Additionally, it is only needed to prove that  $\{\sigma(\mathbf{A}_1), \mathbf{A}_2, \dots, \mathbf{A}_n\}$  are independent, because then we can just apply induction in  $n$ .

We need to check then that if  $A_1 \in \sigma(\mathbf{A}_1), A_2 \in \mathbf{A}_2, \dots, A_n \in \mathbf{A}_n$ , then  $p(A_1 \cap \dots \cap A_n) = p(A_1) \dots p(A_n)$ . Write  $A = A_1, B = A_2 \cap \dots \cap A_n$ . As  $A_2, \dots, A_n$  are independent, we only need to prove that  $p(A \cap B) = p(A)p(B)$ .

Fix now  $B$ , and define  $S(B) = \{A \in \sigma(\mathbf{A}_1) : A \text{ is independent with } B\}$ . We will show that  $S(B)$  is a  $\sigma$ -algebra, and hence  $S(B) = \sigma(\mathbf{A}_1)$ , which is what we want to prove. In particular  $\mathbf{A}_1 \subseteq S(B) \subseteq \sigma(\mathbf{A}_1)$ . Of course  $\emptyset \in S(B)$ . If  $A \in S(B)$  then also  $A^c \in S(B)$ : (13)

write  $\bar{A} = \Omega - A$ . Then  $p(\bar{A} \cap B) = p((\Omega - A) \cap B) = p(\Omega \cap B) - p(A \cap B) = (p(\Omega) - p(A)) p(B) = p(\Omega - A) p(B) = p(\bar{A}) p(B)$ .

We finally need to show  $\sigma$ -additivity: let  $\{C_i\}_{i \in \mathbb{I}}$  a sequence in  $S(B)$ , we want to show that  $\bigcup_{n \geq 1} C_n \in S(B)$ . Observe that if  $B_i$  is independent with  $C_1, \dots, C_r$ , then  $B_i$  is also independent with  $\bigcup_{n=1}^r C_n$ .

Define  $B_n = B \cap \bigcup_{m=1}^n C_m$ . This is an increasing sequence of sets, hence,  $\lim B_n = \bigcup_n B_n = B \cap \bigcup_{n \geq 1} C_n$ . So

$$\begin{aligned} p(B_i \cap \bigcup_{n \geq 1} C_n) &= \lim p(B_i \cap \bigcup_{m=1}^n C_m) = \lim p(B_i) p(\bigcup_{m=1}^n C_m) = \\ &= p(B_i) p(\bigcup_{n \geq 1} C_n), \end{aligned}$$

which proves the  $\sigma$ -additivity, as we wanted to show.

Ob/ We leave as an exercise the following:  $A_1, A_2$  independent with  $A \Leftrightarrow A_1 \cup A_2$  independent with  $A$ . This will be worked in the problem sheet.

We can go back to our problem.

Def/ let  $\{X_i\}_{i \in \mathbb{I}}$  be a family of random variables. Define  $\sigma(\{X_i\})$  as the minimal  $\sigma$ -algebra containing all the events of the form  $\{X_i \leq c\}$ ,  $i \in \mathbb{I}$ ,  $c \in \mathbb{R}$ . We call it the  $\sigma$ -algebra generated by  $\{X_i\}_{i \in \mathbb{I}}$ .

$$\sigma(X_m, X_{m+1}, \dots)$$

let now take  $\{X_n\}_{n \geq 1}$ , and define  $\sigma_m = \sigma(\{X_m, X_{m+1}, \dots\})$ . In particular,  $\sigma_m \supseteq \sigma_{m+1}$ . so we get a decreasing sequence of  $\sigma$ -algebras.

Def/ The tail  $\sigma$ -algebra of  $\{X_n\}_{n \geq 1}$  is defined by  $\bigcap \sigma_m = \lim \sigma_m = \sigma_\infty$ . The elements of  $\sigma_\infty$  are called tail events, and  $\sigma_\infty$ -measurable functions are called tail functions.

Ob/ If we take  $\{A_n\}_{n \geq 1}$ , a sequence of events,  $X_n = \mathbb{I}_{A_n}$ , then  $\sigma_m = \sigma(A_m, A_{m+1}, \dots)$ . So the previous definitions apply as well for sequence of events.

So now we can prove the following result:

Theorem / (Kolmogorov 0-1 law) If  $\{X_n\}_{n \geq 1}$  is independent, then  $\sigma_\infty$  of  $\{X_n\}_{n \geq 1}$  is trivial: each  $A \in \sigma_\infty$  satisfies that  $p(A) = 0$  or  $p(A) = 1$ .

Proof/ let us first proof the following claim:

Claim: If  $\{X_1, \dots, X_m, Y_1, \dots, Y_n\}$  are independent random variables, then  $\sigma(X_1, \dots, X_m)$  and  $\sigma(Y_1, \dots, Y_n)$  are independent.

We denote by  $A$  the set of events of the form  $\{X_1 \leq a_1, \dots, X_m \leq a_m\}$ , and similarly for  $B$ . As  $A$  and  $B$  are independent, because each element from  $A$  is independent from each element from  $B$ , we have then that  $\sigma(A) = \bigcap_{i=1}^m \sigma(X_i)$  and  $\sigma(B) = \bigcap_{j=1}^n \sigma(Y_j)$  are also independent.

Observe now that  $U_m = \bigcup_{i=m+1}^\infty \sigma(X_{m+1}, \dots, X_{i+n})$  is an algebra (possibly NOT a  $\sigma$ -algebra!) so, as  $\{X_1, \dots, X_m\}$  and  $\{X_{m+1}, \dots, X_{m+n}\}$  are independent, then  $\sigma(X_1, \dots, X_m)$  and  $\sigma(X_{m+1}, \dots, X_{m+n})$  are also independent, so  $\sigma(X_1, \dots, X_m)$  and  $\sigma(U_m)$  are independent. As  $\sigma(U_m) = \sigma(X_{m+1}, X_{m+2}, \dots) = \sigma_{m+1}$  we conclude that  $\sigma(X_1, \dots, X_m)$  and  $\sigma_{m+1}$  are ind.

Finally,  $\sigma_\infty \subseteq \sigma_{m+1}$ , so  $\sigma(X_1, \dots, X_m)$  and  $\sigma_\infty$  are independent. Taking  $V = \bigcup_{i=1}^m \sigma(X_i, \dots, X_m)$ , then  $V$  and  $\sigma_\infty$  are independent, and since  $\sigma(V) = \sigma_\infty$  and  $\sigma(\sigma_\infty) = \sigma_\infty$  are independent. Hence,  $\sigma_\infty$  and  $\sigma_\infty$  are independent, which implies that each element  $A \in \sigma_\infty$  satisfies

$$P(A) = P(A \cap V) = P(A)^2 \Rightarrow P(A) \in \{0, 1\}$$