# Problem Sheet 9

**Branching Processes. Random walks.**

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Stochastics II, Summer 2015
Deadline: 23th June 2014 (Tuesday) by 10:00, at the end of the lecture.

`Problem 1 [10 points]:` Let $\{Z_n\}_{n\geq 0}$ be a branching process with offspring distribution $X$ with probability generating function $G(s)$. Each $Z_n$ has also probability generating function $G_n(s)$. Show that if $n < m$ then

$$p(Z_n > N | Z_m = 0) \leq G_m(0)^N$$

`Problem 2 [10 points]:` *Branching process with inmigration.* Assume now the following variant of the usual branching process: each generation of the branching process is augmented by a random number of inmigrants which are indistinguishable from the other members of the population. Assume that the number of inmigrants is independent in each generation and modelled by the probability generating function $H(s)$. Show that the probability generating function of $Z'_n$ (that we write $G_n(s)$) satisfies that $G_{n+1}(s) = G_n(G(s))H(s)$, where as usual $G(s)$ is the probability generating function of the initial offspring $X$.

`Problem 3 [10 points]:` Suppose that a branching process $\{Z_n\}_{n\geq 0}$ is constructed in the following way: it starts with one individual. The individuals in odd and even generations reproduce according to an offspring distribution with generating function $P_1(s)$ and $P_0(s)$, respectively. All independence assumptions are the same as in the classical case.

1. Find an expression for the probability generating function of $Z_n$.
2. Find the equation which defines the probability of extinction.

`Problem 4 [10 points]:` Study Pólya's Recurrence theorem in dimension 1 when the two steps have different probability ($p$ and $1-p$). Is there any significant change?

`Problem 5 [10 points]:`

1. By using the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$, show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.
2. Show that the multinomial coefficient

$$\frac{n!}{j!k!(n-j-k)!}$$

   is maximize when $k$, $j$ and $n-k-j$ are as close $n/3$ as possible (*Hint:* apply some induction argument...)