# Problem Sheet 8

**Tail events. Probability generating functions**

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Deadline: 16th June 2014 (Monday) by 10:00, at the end of the lecture.

`Problem 1 [10 points]:` Let $(\Omega, \mathcal{A}, p)$ a probability space, and $\{X_n\}_{n\geq 1}$ a sequence of random variables. Show that the following events are tail events:

1. $\{\omega \in \Omega : \lim X_n(\omega) \text{ exists}\}$.
2. For each real number $x$, $\{\omega \in \Omega : \limsup X_n(\omega) \leq x\}$.
3. For each real number $x$, $\{\omega \in \Omega : \lim \frac{1}{n} \sum_{r=1}^{n} X_r(\omega) = x\}$.

Justify that in general $\{\omega \in \Omega : \lim \sum_{r=1}^{n} X_r(\omega) = x\}$ is *not* a tail event.

`Problem 2 [10 points]:` Let $X$ be a random variable taking positive integer values, whose probability generating function is $G(s)$. Write $t_n = P(X > n)$. Show that the generating function of the sequence $\{t_n\}_{n\geq q}$ is equal to

$$T(s) = \frac{1 - G(s)}{1 - s}.$$

Additionally, show that $\mathbb{E}[X] = T(1)$ and $\mathrm{Var}[X] = 2T'(1) + T(1) - T(1)^2$.

`Problem 3 [10 points]:` Let $G_1(s)$ and $G_2(s)$ be two probability generating functions of two random variables. If $\alpha \in [0, 1]$, show that $G_1(s)G_2(s)$ and $\alpha G_1(s) + (1 - \alpha)G_2(s)$ also define probability generating functions. Is it true for $G_1(\alpha s)/G_1(\alpha)$ ?

`Problem 4 [10 points]:` Prove that if $\mu = \mathbb{E}[Z_1]$ and $\sigma^2 = \mathrm{Var}[Z_1]$, then

$$\mathrm{Var}[Z_n] = \begin{cases} n\sigma^2, & \mu = 1 \\ \sigma^2(\mu^n - 1)\mu^{n-1}(\mu - 1)^{-1}, & \mu \neq 1. \end{cases}$$

`Problem 5 [10 points]:` Assume that $\mathbb{E}[s^X] = (2 - s)^{-1}$, and $Z_0 = 1$. Let $V_r$ the total number of generations of size $r$. Show that $\mathbb{E}[V_1] = \frac{\pi^2}{6}$.

`Problem 6 [10 points]:` Show that the probability generating function $H_n$ of the total number of individuals in the first $n$ generations satisfies that $H_n(s) = sG(H_{n-1}(s))$.