# Problem Sheet 7

**Borel-Cantelli results and 0-1 Laws**

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Deadline: 9th June 2014 (Monday) by 10:00, at the end of the lecture.

`Problem 1 [10 points]:` Let $(\Omega, \mathcal{A}, p)$ a probability space, and $\{X_n\}_{n\geq 1}$ a sequence of random variables. Show that the set $\{\omega \in \Omega : \lim X_n(\omega) \text{ exists}\}$ is an event (*Hint*: you may want to apply that if $X_n(\omega)$ converges, then the sequence $\{X_n(\omega)\}_{n\geq 1}$ is a Cauchy sequence).

`Problem 2 [10 points]:` *A law of large numbers.* Let $X_1, X_2, \ldots$ be a sequence of independent random variables with the same law. Assume that $\mathbb{E}[X_1] = \mu$, $\mathbb{E}[X_1^2] < +\infty$. Show that

$$\frac{1}{n}\sum_{r=1}^{n} X_r \xrightarrow{2} \mu.$$

(*Comment*: the previous convergence can be strength to be almost surely. This is essentially the so-called *strong law of large numbers*).

`Problem 3 [10 points]:` Let $(\Omega, \mathcal{A}, p)$ be a probability space, and $\{A_n\}_{n\geq 1}$ be a sequence of events such that $\lim p(A_n) = 0$. Show that if

$$\sum_{n\geq 1} p(A_n \cap A_{n+1}^c) < +\infty$$

then $p(\limsup\{A_n\}_{n\geq 1}) = 0$. This gives an alternative version of Borel-Cantelli.

`Problem 4 [10 points]:` Let $(\Omega, \mathcal{A}, p)$ be a probability space. Assume that $A$ and $\{B_i\}_{i=1}^r$ is a set of events. Show that if $A$ is independent with each $B_i$, then $A$ is independent with the their union. Show also that $A$ and $B_i^c$ are independent.

`Problem 5 [10 points]:` Let $\{X_n\}_{n\geq 1}$ be a sequence (not necessarily independent) of random variables with $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] < +\infty$. Show that there exist constants $c_n \to \infty$ such that

$$p\left(\lim \frac{X_n}{c_n} = 0\right) = 1$$

`Problem 6 [10 points]:` Let $r > 2$ and $c > 0$. Show that the set

$$\left\{x \in [0,1] : \left|x - \frac{a}{q}\right| \leq \frac{c}{q^r} \text{ for infinitely many } a, q \in \mathbb{N}_{>0}\right\}$$

has measure 0 (*Hint:* try to write the previous set conveniently in terms of each choice $a, q$, and apply Borel-Cantelli at the end).