Problem Sheet 6

Modes of convergence of Random variables

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Deadline: 2nd June 2014 (Monday) by 10:00, at the end of the lecture.

`Problem 1 [10 points]:` *A bestiary of counterexamples.* We show now that most of the implications not proved in the lectures do not hold.

1. Let X be a Bernoulli random variable taking values 0 and 1 with equal probability $1/2$. Let $X_n = X$ for each n (all these variables are NOT independent). Show that $X_n \xrightarrow{d} X$, $X_n \xrightarrow{d} 1 - X$ but X_n cannot converge to Y in any other mode of convergence.
2. Let $r > s \geq 1$ be positive real numbers. Take a sequence of independent random variables $\{X_n\}_{n\geq 1}$ such that
$$X_n = \begin{cases} n \text{ with probability } n^{-(r+s)/2}, \\ 0 \text{ with probability } 1 - n^{-(r+s)/2}. \end{cases}$$
Show that $X_n \xrightarrow{s} 0$ but $X_n \stackrel{r}{\nrightarrow} 0$.
3. Take an independent sequence $\{X_n\}_{n\geq 1}$ with
$$X_n = \begin{cases} n^3 \text{ with probability } n^{-2}, \\ 0 \text{ with probability } 1 - n^{-2}. \end{cases}$$
Show that $X_n \xrightarrow{p} 0$ but $X_n \stackrel{1}{\nrightarrow} 0$.
4. Let $\{X_n\}_{n\geq 1}$ be an sequence of independent random variables defined by
$$X_n = \begin{cases} 1 \text{ with probability } n^{-1}, \\ 0 \text{ with probability } 1 - n^{-1}. \end{cases}$$
Show that $X_n \xrightarrow{p} 0$ but $X_n \stackrel{a.s.}{\nrightarrow} 0$.

`Problem 2 [10 points]:` Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables, and X another random variable over $(\Omega, \mathcal{A}, p)$. Let C be the event $\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}$. For $\varepsilon > 0$, write $A_n(\varepsilon) = \{\omega \in \Omega : |X_n - X| > \varepsilon\}$, and $A(\varepsilon) = \{\omega \in \Omega : \omega \in A_n(\varepsilon) \text{ infinitely often}\}$. Show that $p(C) = 1$ iff $p(A(\varepsilon)) = 0$.

`Problem 3 [10 points]:` Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that $X_n \xrightarrow{p} X$. Show that there exists a *non-random* increasing sequence of integers $n_1, \ldots$ such that $X_{n_i} \xrightarrow{a.s.} X$ (as $i \to \infty$) (*Hint:* choose a convenient sequence n_i from which we can apply the criteria shown in the lecture for almost sure convergence).

`Problem 4 [10 points]:` Assume that $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, and that all random variables are defined over the same probability space. Show that $X_n + Y_n \xrightarrow{a.s.} X + Y$. Show that the same result happens when dealing with r-th mean convergence mode and convergence in probability, but *not* when dealing with convergence in distribution.

`Problem 5 [10 points]:` Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables and $\{c_n\}_{n\geq 1}$ a sequence of real numbers that converge to c. For all types of convergence of $X_n \to X$, show that the same convergence happen for $c_n X_n$ towards cX (*Hint*: for the convergence in distribution, you may want to use Skorokhod's Representation Theorem).