Problem Sheet 5

Probability Theory Strikes Back

Jun. Prof. Juanjo Rué
Clement Requilé
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Deadline: 26th May 2014 (Monday) by 08:15, at the beginning of the lecture.

`Problem 1 [10 points]:` Let $(\Omega, \mathcal{A}, p)$ be a probability space, and X a random variable. Let $a \in \mathbb{R}$. Show that for each decreasing sequence $\{a_n\}_{n\geq 1}$ with limit a, $F_X(a_n) \to F_X(a)$, and hence $F_X(x)$ is right continuous (*Hint:* consider a a convenient sequence of decreasing events).

`Problem 2 [10 points]:` Prove Markov's inequality. Use the proof of Markov's inequality to prove Tchebicheff's inequality.

`Problem 3 [10 points]:` *Lyapunov's inequality.* Let Z be a random variable over a probability space $(\Omega, \mathcal{A}, p)$ such that $\mathbb{E}[|Z|^r] < +\infty$ for all choice of $r \geq 0$.

- Show that the function $g(r) = \log \mathbb{E}[|Z|^r]$ is convex (*Hint*: conveniently take a pair of random variables X and Y and apply Cauchy-Schwarz).
- Deduce from the previous point that if $a \geq b > 0$ we have that $\mathbb{E}[|Z|^a]^{1/a} \geq \mathbb{E}[|Z|^b]^{1/b}$.

This exercise shows that if Z has finite ath moment, then it also has finite bth moment for all positive $b \leq a$.

`Problem 4 [10 points]:` Let $(\Omega, \mathcal{A}, p)$ be a probability space, $A_1, \dots, A_n$ be events. Show that $A_1, \dots, A_n$ are independent iff $p(B_1 \cap \cdots \cap B_n) = p(B_1) \dots p(B_n)$ where $B_i \in \sigma(A_i) = \{\emptyset, A_i, \overline{A_i}, \Omega\}$.

`Problem 5 [10 points]:` Let X_1 and X_2 be two independent uniform random variables in $[0, 1]$. A stick is broken at distance X_1 and X_2 from one of the ends. Which is the probability that the resulting three pieces can be used to build a triangle?

`Problem 6 [10 points]:` *A counterexample for the covariance.* Given two random variables X, Y, the covariance of X and Y is defined as $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. Then, it is obvious that if X and Y are independent, then $\mathrm{Cov}(X, Y) = 0$. The opposite, however, is not true.

Let X be a $\mathrm{N}(0, 1)$, and ε a random variable independent with X such that $p(\varepsilon = 1) = p(\varepsilon = -1) = 1/2$. Let $Y = \varepsilon X$. Show that the covariance of X and Y is 0, but that they are not independent.