Problem Sheet 4

L_p **spaces**

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Deadline: 19th May 2014 (Tuesday) by 10:00, at the end of the lecture.

Problem 1 [10 points]: *A proof of Young's inequality*: let $p, q \geq 0$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha < 1$ and consider the real differentiable function $\phi(t) = \alpha t - t^\alpha$.

1. Show that $\phi'(t) < 0$ when $0 < t < 1$, $\phi'(t) > 0$ if $t > 1$, $\phi(t) \geq \phi(1)$ and $\phi(t) = \phi(1)$ iff $t = 1$.

2. Use the previous observations to show that $t^\alpha \leq \alpha t + (1 - \alpha)$, $t \geq 0$.

3. Write $\alpha = 1/p$, $t = a/b$ in the previous expression. Manipulate the final form and get Young's inequality as stated in the lecture.

Problem 2 [10 points]: Let (X, χ, μ) be a measure space such that $\mu(X) < +\infty$. Show that

a.- $L_q \subset L_p$ for all $1 \leq p \leq q < \infty$.

b.- $L_r \subset L_p + L_q$ for all $p < r < q$. That is, every $f \in L_r$ can be written as a sum $f_p + f_q$ where $f_p \in L_p$ and $f_q \in L_q$ with the previous condition $p < r < q$.

(*Hint:* In b.−, you would like to write $f \in L_r$ in the form $f = f\mathbb{1}_{\{|f| \geq 1\}} + f\mathbb{1}_{\{|f| < 1\}}$).

Problem 3 [10 points]: for which values of a the following functions are objects in $L_p((0, \infty))$?

a.- x^a.

b.- e^{ax}.

c.- $|\log(x)|^a$.

Problem 4 [10 points]: Is the product of two functions of L_2 in L_1? And is the product of two functions of L_1 in L_1?

Problem 5 [10 points]: Let E be a measurable set with finite measure in (X, χ, μ), $f \in L_2(E)$ and $\{F_n\}_n$ a sequence of measurable sets with $F_n \subset E$ such that $\lim_{n \to \infty} \mu(F_n) = 0$. Show that

$$\lim_{n \to \infty} \frac{1}{\sqrt{\mu(F_n)}} \int_{F_n} |f| d\mu = 0$$

(*Hint:* Firstly, bound conveniently the object under study using Cauchy-Schwartz. Later, use Problem 8 in the Problem Sheet 2.)

Problem 6 [10 points]: Let $1 \leq p_i < \infty$ $(i = 1, 2, 3)$ such that $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$.

1. Show that if f_1, f_2, f_3 are measurable functions over $\mathbb{R}$, then

$$\int_{\mathbb{R}} |f_1 f_2 f_3| \, d\mu \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3}.$$

2. Application: show that if $p < 1/3$, then $(x|1 - x|(2 - x))^{-p} \in L_1(0, 2)$.