# Problem Sheet 3

### The Dominated Convergence Theorem


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Deadline: 12th May 2015 (Tuesday) by 10:00, at the end of the lecture.

`Problem 1 [10 points]:` Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. We define $E = \{x \in [0,1] : f(x) \in \mathbb{Z}\}$.

1. Show that $E$ is measurable.
2. Show that for every choice of $m \in \mathbb{N}$, the function $g : \mathbb{R} \to [0,1]$, defined by $g(x) = |\cos(\pi f(x))|^m$ is measurable.
3. Show that
$$\lim_{m\to+\infty} \int_{[0,1]} g\, d\lambda = \lambda(E).$$

`Problem 2 [10 points]:` Show that if $f \in L(X, \chi, \mu)$ and $g$ is a bounded measurable function, then $fg \in L(X, \chi, \mu)$.

`Problem 3 [10 points]:` Let $\alpha > 1$. Show that
$$\lim_{n\to+\infty} \int_{[0,1]} \frac{nx\sin(x)}{1+(nx)^\alpha}\, d\lambda = 0.$$

`Problem 4 [10 points]:` Compute $\lim_{n\to+\infty} \int_E f_n\, d\lambda$ in the following cases:

1. $f_n(x) = \frac{n\sqrt{x}}{1+n^2x^2}$, $E = [0,1]$,
2. $f_n(x) = \frac{nx}{1+n^2x^2}$, $E = [0,1]$,
3. $f_n(x) = \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)$, $E = [0,+\infty)$.

`Problem 5 [10 points]:` Let $f_n(x) = \frac{-1}{n}\mathbb{I}_{[0,n]}(x)$.

1. Show that the sequence $\{f_n\}_{n\geq 1}$ converge uniformly to $f = 0$ in $[0,+\infty)$.
2. Show that
$$\int_{[0,+\infty)} f_n\, d\lambda = -1, \quad \int_{[0,+\infty)} f\, d\lambda = 0.$$
3. Conclude that
$$\liminf \int_{[0,+\infty)} f_n\, d\lambda < \int_{[0,+\infty)} \liminf f\, d\lambda.$$

Does this example contradict Fatou Lemma?

(Comment: remember the notion of *uniform convergence*: a sequence of functions from $X$ with image in $\mathbb{R}$ $\{f_n\}_{n\geq 1}$ is uniform convergent to $f$ if for all $\varepsilon > 0$ there exists $n \geq n_0(\varepsilon)$ such that for all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$. In other words, the speed of convergence does *not* depend of the choice of $x$).

`Problem 6 [10 points]:` Let $p > a, b > 0$. Compute
$$\int_{[0,+\infty)} \frac{e^{ax} - e^{bx}}{x} e^{-px}\, d\lambda$$
(*Hint*: you may want to study first the derivative with respect to $p$).