# Problem Sheet 2

**Outer measures, the Lebesgue integral and the Monotone Convergence Theorem**

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Deadline: 5th May 2015 (Tuesday) by 8:15, at the beginning of the lecture.

**Problem 1 [10 points]:** Let $X$ be a set. Given an algebra $\mathcal{A}$ over $X$ and a measure $\mu$ over $\mathcal{A}$, remember that the *outer measure* $\mu^*$ of $B \in \mathcal{P}(X)$ is defined as:

$$\mu^*(B) = \inf \sum_{n=1}^{\infty} \mu(E_n),$$

where inf is taken over all families $\{E_n\}_{n \geq 1} \subseteq \mathcal{A}$ such that $B \subseteq \bigcup_{n=1}^{\infty} E_n$. Prove the following properties of $\mu^*$:

1. $\mu^*(\emptyset) = 0$.

2. If $B \in \mathcal{A}$, then $\mu^*(B) = \mu(B)$.

3. If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$.

4. if $\{B_n\}_{n \geq 1} \subseteq \mathcal{P}(X)$ (not necessarily pairwise disjoint) then

$$\mu^* \left( \bigcup_{n=1}^{\infty} B_n \right) \leq \sum_{n=1}^{\infty} \mu^*(B_n).$$

Last property is called *countable subbaditivity*. The proof of Caratheodory Theorem combines this with the Caratheodory's Condition in order to get the equality in case of pairwise disjoint sets.

**Problem 2 [10 points]:** *Invariance property of the Lebesgue measure.* Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set and $E + a$ a translation os this set by $a \in \mathbb{R}$. Show that $E + a$ is also Lebesgue measurable (*Hint*: you may like to use the definition of the Lebesgue measure in terms of outer measure).

**Problem 3 [10 points]:** *A non-measurable set.* As mentioned in the lecture, there are subsets of the reals which are not Lebesgue measurable (in other words, the measure function over them is not defined). Here we build the so-called *Vitali sets*, which are not measurable.

1. For $x, y \in \mathbb{R}$ define $x \sim y$ iff $x - y \in \mathbb{Q}$. Show that $\sim$ is an equivalence relation.

2. Each equivalence class $[x]$ contains points in the interval $[0, 1]$. Let $V \subseteq [0, 1]$ be a set which contains exactly one point from each equivalence class (such set exists due to the Axiom of Choice). For all $q \in \mathbb{Q} \cap [-1, 1]$, define $V_q = V + q$. Show that:

$$[0, 1] \subseteq \bigcup_{q \in \mathbb{Q}} V_q \subseteq [-1, 2],$$

and that $q \neq q'$ then $V_q \cap V_{q'} = \emptyset$.

3. Assume that $V$ is Lebesgue measurable. By Problem 2 $V_q$ is measurable. Get a contradiction from all the previous observations.

**Problem 4 [10 points]:** Let $(X, \mathcal{X}, \mu)$ be a measure space, and let $\psi, \varphi \in \mathcal{M}^+(X, \mathcal{X})$ be simple functions. Show that

$$\int (\psi + \varphi)\, d\mu = \int \psi\, d\mu + \int \varphi\, d\mu.$$

**Problem 5 [10 points]:** We define $a_0 = 0$, $a_n = 1 - (2/3)^n$, and the intervals $F_n = [a_{n-1}, a_n]$ for $n \geq 1$. Let $f(x) = \sum_{n \geq 1} n \mathbb{I}_{F_n}(x)$. Compute

$$\int_{[0,1]} f\, d\lambda.$$

**Problem 6 [10 points]:** Prove the Monotone Convergence Theorem just using Fatou Lemma.

**Problem 7 [10 points]:** Consider the following functions defined in $[0, 1]$:

$$g(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}], \\ 1, & x \in (\frac{1}{2}, 1]. \end{cases}$$
$$f_{2k}(x) = g(x),\ x \in [0, 1]$$
$$f_{2k+1}(x) = g(1 - x),\ x \in [0, 1].$$

Show that $\liminf f_n(x) = 0$ for $0 \leq x \leq 1$, and that $\int_{[0,1]} f_n d\lambda = \frac{1}{2}$. This example shows that Fatou Lemma can be satisfied with the strict inequality.

**Problem 8 [10 points]:** Let $f : X \to \mathbb{R}$ be a measurable function such that $\int_{\mathbb{R}} |f| d\mu < \infty$. Show that for all $\varepsilon > 0$ there exists a value of $\delta := \delta(\varepsilon)$ such that if $\mu(E) < \delta$, then

$$\int_E |f|\, d\mu < \varepsilon.$$

**Problem 9 [10 points]:** Show that Monotone Convergence Theorem does not need to hold for decreasing sequences of functions.