Problem Sheet 10

Moment generating functions and Central Limit theorems

Jun. Prof. Juanjo Rué
Clement Requilé
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Deadline: 1st July 2014 (Wednesday) by 10:00, at the end of the lecture.

`Problem 1 [10 points]:` Find the moment generating functions of the following random variables:

1. Poisson distribution with parameter λ.
2. Uniform distribution in the interval $[a, b]$.
3. The Gamma distribution, with probability density function depending on two parameters k, σ:
$$f(x) = \frac{1}{\Gamma(k)\sigma^k} x^{k-1} e^{-\frac{x}{\sigma}} \mathbb{I}_{[0,+\infty)}(x).$$

`Problem 2 [10 points]:` Let X and Y be random variables taking values at $\{1, \ldots, k\}$. Show that if $M_X(t) = M_Y(t)$ as formal power series (namely, X and Y have the same moments) then X and Y have the same probability distribution.

`Problem 3 [10 points]:` *Cumulant generating function*: The cumulant generating function of a random variable X is defined as

$$K_X(t) = \log \mathbb{E}\left[e^{Xt}\right] = \log(M_X(t)).$$

Observe that $K_X(t)$, as a formal power series in t is well defined as $M_X(0) = 1$. Write $K_X(t) = \sum_{r \geq 0} \frac{1}{r!} \kappa_r(X) t^r$.

1. Express $\kappa_1(X)$, $\kappa_2(X)$ and $\kappa_3(X)$ in terms of the moments of X.
2. Find the cumulants of the normal distribution $N(0, 1)$.
3. Show that if X and Y are independent random variables, then $\kappa_r(X+Y) = \kappa_r(X)+\kappa_r(Y)$. This shows that *cumulants* are very well behaved with respect to the sum of independent random variables.

`Problem 4 [10 points]:` *The Cauchy distribution*: we work now with a probability distribution which serves as an example of random variable that does *NOT* have finite moments, but that we can define its characteristic function. Let C have density function

$$f_C(x) = \frac{1}{\pi(1 + x^2)}.$$

In particular, $\int_{\mathbb{R}} f_C(x)\, dx = 1$.

- Show that $\mathbb{E}[C^r] = +\infty$ for all choice of $r > 0$.
- Show that $\phi_C(t) = e^{-|t|}$ (you can be not 'very rigorous' in this complex integral).

This shows that the characteristic function is defined while the moments are not.

`Problem 5 [10 points]:` Consider a sequence of independent Poisson random variables $X_n = \mathrm{Po}(\lambda_n)$, such that $\lambda_n \to \infty$. Show that

$$\frac{X_n - \mathbb{E}[X_n]}{\sqrt{\mathbb{V}\mathrm{ar}[X_n]}} \xrightarrow{d} N(0, 1).$$