Problem Sheet 1

Preliminaries. σ-algebras and measures

Jun. Prof. Juanjo Rué
Clement Requilé
Stochastics II, Summer 2015
Deadline: 28th April 2015 (Tuesday) by 10:00, at the end of the lecture.

`Problem 0 [0 points]:` (**Mandatory**) When presenting the solutions of the first problem sheet, you must write also:

- Your complete name and matriculation number (specially if you are not from FU Berlin).
- An e-mail which I can use to contact you.

`Problem 1 [10 points]:` Let $\{A_n\}_{n\geq 1}$ a sequence of subsets in a certain set X. Define

- $\limsup\{A_n\}_{n\geq 1} = \bigcap_{m\geq 1}\bigcup_{n\geq m} A_n$ is the set of elements of X which belongs to infinite sets of the sequence $\{A_n\}_{n\geq 1}$.
- $\liminf\{A_n\}_{n\geq 1} = \bigcup_{m\geq 1}\bigcap_{n\geq m} A_n$ is the set of elements of X which belongs to all but except a finite number of the sets of the sequence $\{A_n\}_{n\geq 1}$.

Prove that

1. If $\{A_n\}_{n\geq 1}$ is an monotone increasing sequence of sets, then $\limsup\{A_n\}_{n\geq 1} = \bigcup_{n\geq 1} A_n = \liminf\{A_n\}_{n\geq 1}$.
2. If $\{A_n\}_{n\geq 1}$ is an monotone decreasing sequence of sets, then $\limsup\{A_n\}_{n\geq 1} = \bigcap_{n\geq 1} A_n = \liminf\{A_n\}_{n\geq 1}$.

`Problem 2 [10 points]:` Show that every open set in $\mathbb{R}$ is a countable union of open intervals (one can prove it with or without the Axiom of Choice!). This shows that Borel sets are also generated by open sets.

`Problem 3 [10 points]:` Show that

$$[a,b] = \bigcap_{n=1}^{\infty}\left(a-\frac{1}{n}, b+\frac{1}{n}\right), (a,b) = \bigcup_{n=1}^{\infty}\left[a+\frac{1}{n}, b-\frac{1}{n}\right].$$

This shows that every σ-algebra containing closed intervals also contains open intervals, and viceversa. Show also that the Borel algebra is also generated by semiopen intervals $(a,b]$ and sets of the form (a,∞).

`Problem 4 [10 points]:` Let $\mathcal{X}$ be a σ-algebra over X, f be a $\mathcal{X}$-measurable function from X to $\mathbb{R}^*$ and $n>0$. Define

$$f_n(x) = \begin{cases} f(x), \text{ if } |f(x)|\leq n, \\ n, \text{ if } f(x)>n, \\ -n, \text{ if } f(x)<-n. \end{cases}$$

Show that f_n is a $\mathcal{X}$-measurable.

`Problem 5 [10 points]:` Give examples of measurable spaces $(X,\mathcal{X})$ and functions $f: X\to\mathbb{R}$ which are not $\mathcal{X}$-measurable, but such that $|f|$ and f^2 are $\mathcal{X}$-measurable.

`Problem 6 [10 points]:` Let $\mathcal{X}$ be a σ-algebra over X. Show that a function $f : X \to \mathbb{R}$ is $\mathcal{X}$-measurable if and only if $A_\alpha = \{x \in X : f(x) > \alpha\} \in \mathcal{X}$ for all $\alpha \in \mathbb{Q}$. (*Hint:* you may want to use the well known fact that every real number can be approximated by a sequence of rational numbers).

`Problem 7 [10 points]:` Let $\mathcal{X}$ be a σ-algebra over X, $f : X \to \mathbb{R}$ a $\mathcal{X}$-measurable function and $\phi : \mathbb{R} \to \mathbb{R}$ a continous function. Show that $\phi \circ f$ is $\mathcal{X}$-measurable (*Hint:* you may want to use the result proven in Problem 2).

`Problem 8 [10 points]:` Let $\mathcal{X}$ be a σ-algebra over X. Show that it μ_1, μ_2, $\ldots, \mu_n$ are measures over $\mathcal{X}$, and $a_1, \ldots, a_n$ are non-negative real numbers, then the linear combination $\sum_{j=1}^{n} a_j \mu_j$ is a measure over $\mathcal{X}$.

`Problem 9 [10 points]:` Let $(X, \mathcal{X}, \mu)$ be a measure space, and $\{E_n\}_{n\geq 1}$ a sequence of sets in $\mathcal{X}$. Assuming that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) < +\infty$, show that $\limsup\{\mu(E_n)\}_{n\geq 1} \leq \mu(\limsup\{E_n\}_{n\geq 1})$. Show that this is not true if we assume that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = +\infty$ (You may want to remember which is the definition of $\limsup\{E_n\}_{n\geq 1}$).