SOLUTIONS TO PROBLEMS 4 AND 5

Problem 4 (2 points): Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables.

(1) Show that for all $\varepsilon > 0$,
$$P(|X_n| > \varepsilon) \leq \frac{1+\varepsilon}{\varepsilon}\mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] \text{ (1 points).}$$

(2) Show that $X_n \xrightarrow{p} 0$ if and only if $\lim_{n\to\infty}\mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] = 0$ (1 points).

Solution: We start proving 1.- To do so, observe that:
$$|X_n| > \varepsilon \Leftrightarrow |X_n| + 1 > 1 + \varepsilon \Leftrightarrow |X_n| + \varepsilon|X_n| > \varepsilon + \varepsilon|X_n|,$$
and as $1 + |X_n| > 0$, this last inequality is equivalent to $\frac{|X_n|}{1+|X_n|} > \frac{\varepsilon}{1+\varepsilon}$. So,
$$P(|X_n| > \varepsilon) = P\left(\frac{|X_n|}{1+|X_n|} > \frac{\varepsilon}{1+\varepsilon}\right),$$
because they are the same event. Now, we can just apply Markov's inequality because the random variable $\frac{|X_n|}{1+|X_n|}$ takes only positive values:
$$P\left(\frac{|X_n|}{1+|X_n|} > \frac{\varepsilon}{1+\varepsilon}\right) \leq \frac{1+\varepsilon}{\varepsilon}\mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right]$$
Let us go now to prove 2.- Observe that the implication $\Leftarrow$ is immediate from what we have done at point 1.-: if $\mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right]$ tends to 0, then
$$0 \leq P(|X_n| > \varepsilon) = P\left(\frac{|X_n|}{1+|X_n|} > \frac{\varepsilon}{1+\varepsilon}\right) \leq \frac{1+\varepsilon}{\varepsilon}\mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] \to 0$$
and so $X_n \xrightarrow{p} 0$. To prove the inverse implication $\Rightarrow$, observe that $\frac{|X_n|}{1+|X_n|} \leq 1$, so for every $\varepsilon > 0$ we have that
$$\mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] \leq 1 \times P\left(\frac{|X_n|}{1+|X_n|} \geq \frac{\varepsilon}{1+\varepsilon}\right) + \frac{\varepsilon}{1+\varepsilon}P\left(\frac{|X_n|}{1+|X_n|} < \frac{\varepsilon}{1+\varepsilon}\right)$$
and so, this can be written as $P\left(|X_n| \geq \varepsilon\right) + \frac{\varepsilon}{1+\varepsilon}P\left(|X_n| < \varepsilon\right)$. If $X_n \xrightarrow{p} 0$, then we conclude that
$$\mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] \leq P\left(|X_n| \geq \varepsilon\right) + \frac{\varepsilon}{1+\varepsilon}P\left(|X_n| < \varepsilon\right) \to \frac{\varepsilon}{1+\varepsilon}.$$
As this convergence is true for every choice of $\varepsilon > 0$, making $\varepsilon \to 0$ we have the result as claimed.

Problem 5 (2 points): For f, $g \in L^1(\mathbb{R})$, we define

$$(f * g)(x) = \int_{\mathbb{R}} f(t)g(x-t)\,dx$$

We will show in this problem that there does not exist an identity element $\delta \in L^1(\mathbb{R})$ such that $\delta * f = f * \delta = f$ for all $f \in L^1(\mathbb{R})$. Assume its existence.

(1) Show that if E has finite measure, then
$$\int_E \delta(x)\,dx = \begin{cases} 1, & 0 \in E, \\ 0, & 0 \notin E. \end{cases} \qquad \text{(0.5 points)}$$
(2) Write $E' = \{x \in \mathbb{R} : \delta(x) > 0\}$. Show that $\int_{E'} \delta(x)\,dx = 0$. Prove a similar result for $F' = \{x \in \mathbb{R} : \delta(x) < 0\}$. (0.75 point).
(3) Conclude from the previous points that $\delta(x) = 0$ for almost all x, and get a contradiction from this fact (0.75 points).

We start with 1.-, if E has finite measure, then its indicator function $\mathbb{I}_E(x) \in L^1(\mathbb{R})$. Hence,
$$\int_E \delta(t)\,dt = \int_{\mathbb{R}} \mathbb{I}_E(t)\delta(t)\,dt = \int_{\mathbb{R}} \mathbb{I}_{-E}(0-t)\delta(t)\,dt = \mathbb{I}_{-E}(0)$$
where $-E = \{-x \in E\}$. This is true because E has finite measure if and only if $-E$ has finite measure. Finally $\mathbb{I}_{-E}(0) = \mathbb{I}_E(0)$.

Now let us go to point 2.- The main difficulty here is that we cannot apply directly point 1.- , because we do not know if E' has finite measure. We may assume that $\delta(0) = 0$ (otherwise we can just redefine it satisfying this, which only makes a difference in a set of measure 0). To do so, we approximate it by the set $E_n = \{x \in \mathbb{R} : 0 < |x| \le n, \delta(x) > 0\}$, which has finite measure. In particular, $\{E_n\}_{n\ge1}$ is an increasing sequence of sets, with limit $E' = \bigcup_{n\ge1} E_n$. Observe that each function $\delta(x)\mathbb{I}_{E_n}(x)$ is dominated by $|\delta(x)| \in L^1(\mathbb{R})$. We can apply then the Dominated Convergence Theorem with catalytic function $g(x) = \delta(x)$. So:
$$\int_{E'} \delta(x)\,dx = \int_{\mathbb{R}} \lim \delta(t)\mathbb{I}_{E_n}\,dt = \lim \int_{\mathbb{R}} \delta(t)\mathbb{I}_{E_n}(t)\,dt = 0$$
because the integral over E_n, by point 1.- is equal to 0. A similar argument applies when dealing with F'.

To conclude, we split $\mathbb{R}$ in terms of E', F' and the set $A = \{x \in \mathbb{R} : \delta(x) = 0\}$, so
$$\int_{\mathbb{R}} \delta(x)\,dx = \int_{E'} \delta(x)\,dx + \int_{F'} \delta(x)\,dx + \int_A \delta(x)\,dx = 0 + 0 + 0 = 0,$$

where the last 0 holds because the function is equal to 0 over 0. So we have that $\delta(x)$ is 0 almost always. We conclude the argument taking an arbitrary function $0 \neq f \in L^1(\mathbb{R})$, then, by the definition:

$$f(x) = (\delta * f)(x) = \int_{\mathbb{R}} \delta(x) f(x-t)\, dx = 0$$