Mock Exam

Stochastics II, Summer 2015
Jun. Prof. Juanjo Rué
Clément Requilé
Deadline: 8th July 2015 (Wednesday) by 10:00, at the end of the lecture.
No points are given to the final grading

Problem 1 (2 points):

- Let (X, χ, μ) a measurable space. Define χ-measurable function over (X, χ, μ) (0.5 points).
- Let (X, χ, μ) a measurable space. the set $L(X, \chi, \mu)$ of integrable functions (0.5 points).
- Prove that $f \in L(X, \chi, \mu)$ if and only if $|f| \in L(X, \chi, \mu)$ (1 point).

Problem 2 (2 points): Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables.

- Define convergence in distribution and convergence in the r-th mean mode (0.5 points).
- Prove that if $X_n \xrightarrow{d} X$ and $p(|X_n| \leq k) = 1$ for all n and some k, then $X_n \xrightarrow{r} X$, $r \geq 1$ (1.5 points).

Problem 3 (2 points): Let $f_n(x) = \frac{-1}{n}\mathbb{I}_{[0,n]}(x)$.

- Show that the sequence $\{f_n\}_{n\geq 1}$ converge uniformly to $f = 0$ in $[0, +\infty)$ (0.5 points).
- Show that
$$\int_{[0,+\infty)} f_n \, d\lambda = -1, \quad \int_{[0,+\infty)} f \, d\lambda = 0 \qquad (0.5\,\text{points}).$$
- Conclude that
$$\liminf \int_{[0,+\infty)} f_n \, d\lambda < \int_{[0,+\infty)} \liminf f \, d\lambda \quad (0.5\,\text{points}).$$
- Does this example contradict Fatou Lemma? (0.5 points)

Problem 4 (2 points):

- Find the moment generating function of the Poisson distribution with parameter λ (1 point).
- Show that a sequence of binomial random variables $\text{Bin}(n, \frac{\lambda}{n})$ converges in distribution to a Poisson random variable with parameter λ (1 point).

Problem 5 (2 points): Let $f \in L^p((0,1))$ for all $p \geq 1$, such that $||f||_{r+1}^{r+1} \leq r||f||_r^r$. Show that for $0 \leq \alpha < 1$ we have that
$$\int_{(0,1)} e^{\alpha|f(x)|} \, dx \leq 1 - ||f||_1 \log(1 - \alpha).$$

- The time for this exam is 2 hours.
- You can solve each section (or subsection) independently.
- You should try to write and justify ALL steps.

- This will NOT be counted towards your homework requirements.