Make up Exam

Stochastics II, Summer 2015
Jun. Prof. Juanjo Rué
Clément Requilé

Problem 1 (2 points):

- Define convergence almost surely, in probability and in distribution for sequences of random variables (1 point).
- Prove that convergence in probability implies convergence in distribution (1 point).

Problem 2 (2 points): State and prove Borel-Cantelli Lemma.

Problem 3 (2 points): Let X be a random variable taking positive integer values, whose probability generating function is $G(s)$. Write $t_n = P(X > n)$. Show that the generating function of the sequence $\{t_n\}_{n\geq 0}$, $T(s) = \sum_{n\geq 0} t_n s^n$ is equal to

$$T(s) = \frac{1 - G(s)}{1 - s}.$$

Additionally, show that $\mathbb{E}[X] = T(1)$ and $\mathrm{Var}[X] = 2T'(1) + T(1) - T(1)^2$.

Problem 4 (2 points): Let $\mathbf{X}$ be a time-homogenous Markov chain with N states with transition probability matrix P. We say that the transition matrix is *doubly stochastic* if both its rows and columns sum 1 (this is stronger than the usual condition for P).

1. Show that, for each $n \geq 0$, the matrix P^n is doubly stochastic, and that $\pi = (N^{-1}, \ldots, N^{-1})$ is a stationary distribution (1 point).
2. Show that all states are non-null recurrent (1 point).

Problem 5 (2 points): Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2}, \ 0 < x < 1, \\ 0, \ \text{otherwise}. \end{cases}$$

1. Show that $f \in L^1(\mathbb{R})$ (0.75 points).
2. Let $\{r_n : r_n \in \mathbb{Q}\}$ a fixed ordering of the rational numbers. Define

   $$g(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x - r_n).$$

   Show that $g \in L^1(\mathbb{R})$ (0.75 points).
3. Show that g is unbounded on any open interval, despite being integrable (0.5 points).

- The time for this exam is 2 hours.
- You can solve each section (or subsection) independently.
- You should try to write and justify ALL steps.
- The preliminary grading of the subject will be available on the 18th September both on the webpage and in my office.
- You can come to my office on Monday 21st September from 09:00 to 11:00 to see the exam.