Bonus Problems

Extra Problems on Enumerative Techniques in Combinatorics

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Discrete Mathematics III, Summer 2014
Deadline: 23th June 2014 (Monday) by 14:00, at the end of the lecture.

This is some kind of mixture of all the enumerative techniques we have encountered so far. You would like to write at most two solutions in order to be corrected. This corrections won't contribute to the total score of the homework.

`Problem 1 :` *Still another way to count permutations:* consider the class P of all permutations written as words over integers starting from 1 (so, all the generating functions we are considering are ordinary). One can go from a permutation of size $n-1$ to a permutation of size n by selecting a gap and inserting the value n. Writing $P(x)$ for the generating function of permutations, show, by encoding the previous fact, that

$$P(x) = 1 + x\frac{d}{dx}(xP(x)),$$

and conclude that $[x^n]P(x) = n[x^{n-1}]P(x)$.

`Problem 2 :` *Dissections of a polygon:* A *dissection* of a labelled polygon (remember: we enumerate vertices from 1 to n following the contourn of the polygon) is a decomposition of the interior of the polygon into two dimensional regions, where the degree of each internal face is not restricted (it must be greater or equal than 3).

Prove that the generating function $D(x,u)$ which counts the number of dissections of a polygon, where x, u marks vertices and faces, respectively, satisfies the relation:

$$(1+u)D(x,u)^2 - x(1+x)D(x,u) + x^3 = 0.$$

`Problem 3 :` *Labelled hierarchies:* the class of *labelled hierarchies* is formed of trees whose internal vertices are unlabelled and are constrained to have outdegree larger than 1, while their leaves have labels attached to them. As for other labelled structures, size is the number of labels (internal vertices do not contribute). Writing $L(x)$ for its exponential generating function, show that

$$L(x) = x + e^{L(x)} - 1 - L(x).$$

Prove that $L(x) = T\left(\frac{1}{2}e^{x/2-1/2}\right) + \frac{x}{2} - \frac{1}{2}$, where $T(x)$ is the generating function of rooted labelled trees.

`Problem 4 :` *Graphs with fixed excess:* we have seen already how to count unicyclic graphs (i.e., graphs with exactly one cycle). The *excess* of a connected graph G is equal to $e(G) - v(G)$ (the number of edges minus the number of vertices). Hence, trees are the connected graphs with excess equal to -1, and unicyclic connected graphs are the ones with excess equal to 0.

- Write the GF for labelled connected graphs with fixed excess 2 and 3.
- Can you state a general result on the form of the GF of connected graphs with fixed excess? (*Hint:* is a rational function on the Cayley function $T(x)$).
- At which excess the GF for connected graphs is different from the GF for connected *planar* graphs?