Problem Sheet 9

More on map enumeration

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Discrete Mathematics III, Summer 2014
Deadline: 14th July 2014 (Monday) by 14:00, at the end of the lecture.

`Problem 1 [10 points]:` *Bipartite maps:* A rooted planar map is *bipartite* if all its cycles have even length. Write $B(x, u)$ the generating function of bipartite rooted planar maps, where x marks edges and u marks the length of the root face.

Find (i.e., draw!) all rooted bipartite planar maps with $n = 0, 1, 2, 3$ edges. Finally, by means of the root decomposition, get the equation that $B(x, u)$ satisfies, possibly in terms of $B(x, 1)$.

`Problem 2 [10 points]:` *Maps on surfaces*: most of the things we are doing for *planar* maps can be generalized in surfaces, but one needs to be more careful. The definition is almost the same, but now we require that each two dimensional region is homeomorphic to a disk (observe for example that a single edge embedded in the torus does *not* define a map on the torus).

Exhibit an example showing that the property concerning the distances in quadrangulations (namely, that two adjacent vertices in a quadrangulation do not have the same distance to the origin) does not hold in surfaces.

`Problem 3 [10 points]:` *2-connected rooted objects*: By means of the root decomposition we will find the equations for 2-connected triangulations (namely, all faces have degree 3, there is not a cutvertex). In order to do so, we consider *pseudo-triangulations*: those are either the single (rooted) edge or a 2-connected rooted map where the root face (that is, the unbounded face) is not a triangle (but the rest are).

1. Use variable x to mark edges and u the degree of the unbounded face. Denote by $T(x, u)$ the corresponding GF and $F(x) = [u^2]T(x, u)$. Show that:

$$T(x, u) = xu^2 + \frac{x}{u}T(x, u)^2 + \frac{x}{u}(T(x, u) - u^2 F(x)).$$

   (*Hint:* in the last two terms, you should consider the cases where the resulting map is wether 2-connected or not...) Indeed, (yo do not have to prove the following...) in this case we are also able to apply the Quadratic Method in order to get $u = \theta(x)$, and working a little more gets that

$$[x^{3n+1}]F(x) = \frac{2^{n+1}}{(2n+1)(2n+2)}\binom{3n}{n}.$$

2. Extend the previous ideas in order to get the equations for enumerating 2-connected rooted quadrangulations.