Problem Sheet 8

Catalytic variables and map enumeration

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Discrete Mathematics III, Summer 2014
Deadline: 7th July 2014 (Monday) by 14:00, at the end of the lecture.

`Problem 1 [10 points]:` *The Kernel method in Dyck paths:* we consider (again) the enumeration of Dyck paths but now we will use a catalytic variable. Instead of considering Dyck paths, we study *pseudo*-Dyck paths: those are the paths starting at $(0, 0)$, with steps $(1, 1)$ and $(1, -1)$ such that they do not cross (but can touch) the axis $y = 0$, but now we do not require that the path finishes at $y = 0$ (it can finish at any level).

Write $F(t, u)$ for the GF of pseudo-Dyck paths where t marks the number of steps and u marks the height of the final point on the path.

1. Show that

$$F(t, u) = 1 + tuF(t, u) + \frac{t}{u}(F(t, u) - F(t, 0)).$$

2. Use the fact that we know that $F(t, 0) = \frac{1-\sqrt{1-4t^2}}{2t^2}$ in order to get $F(t, u)$.

3. Write the first equation in the form $F(t, u)g(t, u) = 1 + h(t, u)F(t, 0)$. As we have a level of freedom, write $u := \theta(t)$ and make $g(t, \theta(t)) = 0$. From this, get expressions for $F(t, 0)$ and $F(t, u)$ without assuming knowledge of $F(t, 0)$, and check that you get again the previous expression for $F(t, 0)$.

The main difference in the method is that here the expression is linear, and in the case of maps was quadratic. In our situation now we call the method *the kernel method*.

`Problem 2 [10 points]:` *Non-separable rooted maps in terms of rooted maps:* a map is *non-separable* if the the corresponding multigraph is 2-connected (it does not have cutvertices). Writing $N(x)$ the GF for rooted non-separable maps, show that:

$$M(x) = N(x(1 + M(x))^2),$$

where $M(x) := M(x, 1)$ is the GF obtained in the lectures for rooted maps (*Hint:* pay attention at the *corners* of the map...).

With a little more of work one can derive an implicit expression for $N(x)$.

`Problem 3 [10 points]:` *Maps with minimum degree greater or equal than* 2*:* Let $D(x)$ the generating function of maps whose minimum degree is greater or equal than 2. Write a relation between $D(x)$ and $M(x)$ (*Hint:* use the ideas of pruning as we did in graphs, but here be careful with the corners of the map...)