Problem Sheet 7

Subexponential growth and the Transfer Theorem

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Discrete Mathematics III, Summer 2014
Deadline: 30th June 2014 (Monday) by 14:00, at the end of the lecture.

Note: Problem 1 is compulsory for all. Hence, you should do Problem 1 and one of the rest of the problems.

`Problem 1 [10 points]:` *A way to remember Stirling formula:* We proceed now to make the asymptotic study of the equation $T(z) = ze^{T(z)}$.

1. Show that the hypothesis for the Lagrangian scheme are satisfied.
2. Show that the smallest singularity of $T(z)$ is located at $z = e^{-1}$.
3. Writing $T(x) = t_0 + t_1 X + O(X^2)$ (where $X = (1 - ex)^{1/2}$), prove that $t_0 = 1$ and $t_1 = -\sqrt{2}$.
4. Apply the Transfer Theorem in order to get the asymptotic estimate for the number of rooted labelled trees.
5. Finally, applying that this number is equal to n^{n-1}, rediscover Stirling formula.

(In fact, Stirling formula is hidden inside the Transfer Theorem, but is this a good way to recall why we have a behaviour like n^n in labelled trees).

`Problem 2 [10 points]:` *The analytic interpretation of the Dissymmetry Theorem*: we will now study the singular expansion of $U(x) = T(x) - \frac{1}{2}T(x)^2$.

1. Using the singular expansion in Problem 1, show that the singular expansion of $U(x)$ is of the form $u_0 + 0X + ...$ (where $X = (1 - ex)^{1/2}$).

This means that there is a cancelation of the singular terms. Hence, we need more terms in the expansion of $T(x)$

2. Show that $T(x) = t_0 + t_1 X + t_2 X^2 + t_3 X^3 + O(X^4)$ (where $X = (1 - ex)^{1/2}$), where $t_2 = \frac{2}{3}$ and $t_3 = -\frac{11}{36}\sqrt{2}$.
3. Using now this expansion, show that the singular expansion of $U(x)$ is $\frac{1}{2} - X^2 + \frac{2\sqrt{2}}{3}X^3 + O(X^4)$.
4. Use the Transfer Theorem to get asymptotic estimates and compare this estimate with the number of Cayley trees (n^{n-2}) (*Hint:* $\Gamma(-3/2) = \frac{4}{3}\sqrt{\pi}$).

Roughly speaking, this tells us that the Dissymmetry Theorem provides an analytic cancelation.

`Problem 3 [10 points]:` *Some meromorphic families* Study the singular expansion of the following combinatorial families:

1. The EGF for derangements: $D(z) = \frac{e^{-z}}{1-z}$.
2. The EGF for permutations without cycles of length 1 or 2: $E(z) = \frac{e^{-z-z^2/2}}{1-z}$.

Generalize this results by obtaining asymptotic estimates for the number of permutations without cycles of lengths in a set $\Delta \in \mathbb{N}$.

`Problem 4 [10 points]:` *Universality of tree enumeration:* Let Δ be a subset (possibly infinite) of $\mathbb{N}$, $0 \in \Delta$, and consider rooted embedded trees where the outdegree of the vertices is a value in Δ (in particular, leaves of the tree have outdegree 0).

1. Let $T_\Delta(z)$ the associated GF, where z marks the total number of vertices. Show that $T_\Delta(z) = z \sum_{\delta \in \Delta} T_\Delta(z)^\delta$.
2. Show that the equation satisfies the conditions of the Lagrangian scheme.
3. Conclude that the subexponential growth is *universal* for the class of rooted labelled trees, and is always equal to $n^{-3/2}$ (here a qualitative argument is enough).

In order to get the precise asymptotic in this case, we should know if the set Δ is periodic or not (because then multiple singularities may arise).