Problem Sheet 6

Analytic techniques on generating functions

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Discrete Mathematics III, Summer 2014
Deadline: 16th June 2014 (Monday) by 14:00, at the end of the lecture.

`Problem 1 [10 points]:` *Change-making:* Assume that we live in a country with coins of value $a_1, a_2, \ldots, a_r$, where $\gcd(a_1, \ldots, a_r) = 1$.

1. Write the GF associated to the number of solutions to the equation $\sum_{i=1}^{r} a_i x_i = n$ $(0 < x_i)$ (namely, the number of ways in which we can change make using the coins in our country). Conclude that the GF you obtain is rational, and that it is analytic at $z = 0$.
2. Describe how the singularities of this GF are.
3. Conclude that there exists an N_0 such that for all $n \geq N_0$, the number of solutions of $\sum_{i=1}^{r} a_i x_i = n$ with $0 < x_i$, $i = 1 \ldots r$ is strictly greater than 0. Hence, for n large enough we can always change make (*Hint:* you may write the corresponding GF in a canonical way...).

`Problem 2 [10 points]:` *Generalized Catalan numbers:* Remember that decompositions of a labelled polygon into $(k+1)$-agons are counted using the generating function $C_k(x)$, which satisfies that $C_k(x) = 1 + xC_k(x)^k$, where x marks $(k+1)$-agons.

1. Apply the results developed in the lecture in order to get the exponential order of the coefficients.
2. Check the previous result by direct computation over the exact expressions and using the Stirling approximation formula.

`Problem 3 [10 points]:` *Alignments:* An *alignment* is an ordered sequence of labelled cycles.

1. Write the combinatorial specification for this family, and the corresponding EGF.
2. Get directly from the previous expression the exponential order of the sequence counting alignments.

`Problem 4 [10 points]:` *Representation functions:* let A be an infinite subset of the positive integers. Write $r_A(n) = |\{a + a' = n,\, a, a' \in A\}|$. Our objective in this problem is to show the following: there do NOT exist an N_0 such that if $n \geq N_0$ then $r_A(n)$ is constant.

1. Prove directly this result by using a parity argument. (*Hint*: which is the parity of $r_A(2a)$, for $a \in A$?)
2. Assume the contrary (namely, there is an N_0 such that $r_A(n) = c$ for $n \geq N_0$). Writing $A(z) = \sum_{a \in A} z^a$, show that:
$$A(z)^2 = \frac{g(z)}{1 - z}$$
where $g(z)$ is a polynomial satisfying that $g(1) \neq 0$.
3. Consider now the representation function $R_A(n) = |\{a + a' = n,\, a, a' \in A, a \leq a'\}|$. Here the same statement cannot be proven using elementary arguments. Write now the equations encoding this representation function and by a complex analytic argument conclude a contradiction (*Hint*: you may wonder what happen when $z \to -1$ in the region of convergence...)

An open, difficult and long-standing question is the following conjecture of Erdős and Turán: if $r_A(n) > 0$ for $n \geq N_0$, then the sequence $\{r_A(n)\}_n$ cannot be bounded (this is the so-called *Erdős-Turán Conjecture*).