Problem Sheet 5

More on Exponential Generating Functions

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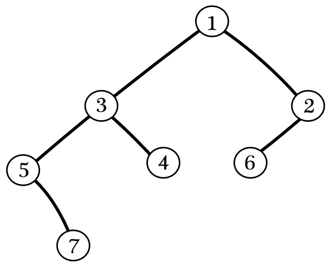
Discrete Mathematics III, Summer 2014
Deadline: 2nd June 2014 (Monday) by 14:00, at the end of the lecture.

`Problem 1 [10 points]:` *The boxed product of labelled combinatorial classes:* let $\mathcal{B}$ and $\mathcal{C}$ be labelled combinatorial classes, with corresponding EGF $B(x)$ and $C(x)$. Assume that $[x^0]B(x) = 0$. The *boxed product* of $\mathcal{B}$ and $\mathcal{C}$ is denoted by $\mathcal{B}^{\square} * \mathcal{C}$, and is the subset of the labelled product of $\mathcal{B}$ and $\mathcal{C}$ formed with elements such that the smallest label is constrained to lie in the $\mathcal{B}$ component.

1. Show that the corresponding EGF is

$$\int_0^x C(x)\frac{d}{dx}B(x)dx.$$

2. To each permutation, one can associate bijectively a binary tree of a special type called an *increasing binary tree.* This is a plane rooted binary tree in which internal nodes bear labels in the usual way, but with the additional constraint that node labels increase along any branch stemming from the root. See the following example for the permutation 5734162:



Denoting by $\bullet$ the atom of size 1, by ϵ the object of size 0 and by $\mathcal{I}$ the set of increasing binary trees (where the size is the number of vertices), prove that $\mathcal{I} = \{\epsilon\} \cup \bullet^{\square} * (\mathcal{I} * \mathcal{I})$. Conclude that the associated EGF is $I(x) = (1-x)^{-1}$, and rediscover that the number of permutations of size n is equal to $n!$.

`Problem 2 [10 points]:` *The number of connected graphs:* Get the bivariate EGF for labelled graphs (where x marks vertices and y marks edges), and obtain the EGF for connected labelled graphs in terms of the previous one (*Comment:* here it is not expected that you get closed formulas, because such formulas do not exist. Get the first EGF as an infinite sum, and argue how is the EGF for connected graphs in terms of the first one).

`Problem 3 [10 points]:` *Permutations with square root:* let a_n be the number of permutations $\sigma \in \mathfrak{S}_n$ that have a square root, namely, there exists $u \in \mathfrak{S}_n$ satisfying $u^2 = \sigma$.

1. A permutation σ has an square root if and only if σ decomposes into a product of disjoint cycles where the number of cycles of each even length $2i$ is even.

2. Using the previous fact, show that EGF is equal to

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} = \left(\frac{1+x}{1-x}\right)^{1/2} \prod_{k\geq 1} \cosh\left(\frac{x^{2k}}{2k}\right).$$

`Problem 4 [20 points]:` *Graph decompositions and Cacti graphs:*

In this problem we will see how to translate structural facts on graphs into enumerative formulas. Consider a family of *connected* labelled graphs $\mathcal{C}$ defined from a set of 2-connected labelled graphs $\mathcal{B}$ (namely, a graph in $\mathcal{C}$ is built by using the objects of $\mathcal{B}$).

1. Show that two blocks of a connected graph are disjoint or just intersect on one vertex. Conclude that there exists a tree-like structure relating the blocks of a graph and cutvertices.

2. Consider the family of vertex *rooted* connected graphs $\mathcal{C}^\bullet$. Explain that we have the following combinatorial description:

$$\mathcal{C}^\bullet = \bullet * \mathrm{Set}(\mathcal{B}^\circ(\mathcal{C}^\bullet)),$$

   where $\mathcal{B}^\circ(\mathcal{C}^\bullet)$ is the combinatorial class constructed from $\mathcal{B}^\circ$ by substitution of each vertex by an element in $\mathcal{C}^\bullet$. Conclude that the GFs are related via the equation:

$$xC'(x) = x\exp(B'(xC'(x))),$$

   which gives the coefficients of $xC'(x)$ (and consequently the ones of $C(x)$) by applying Lagrange's inversion formula.

3. A labelled *Cacti graph* is a connected graph whose blocks are cycles. Get the expression satisfied by the EGF of vertex rooted labelled Cacti graphs.

Later in the course we will see how to analyze such equation from an analytic point of view.

`Problem 5 [10 points]:` *Some computations:*

1. Let σ be a uniformly at random permutation of size n, and denote by $X_n(\sigma)$ the number of cycles of σ. Compute $\mathbb{E}[X_n]$, $\mathbb{E}[X_n^2]$, and conclude the value for $\mathrm{Var}[X_n]$.

2. Let t be a uniformly at random rooted embedded unlabelled tree of size n. Denote by $X_n(t)$ the outdegree of the root. Compute the corresponding bivariate GF and extract the expected value for X_n.