Problem Sheet 4

Exponential Generating Functions

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Discrete Mathematics III, Summer 2014
Deadline: 26th May 2014 (Monday) by 14:00, at the end of the lecture.

`Problem 1 [10 points]:` Let $\mathcal{A}$ be a labelled combinatorial family and denote by $\widehat{\mathcal{A}}$ the corresponding unlabelled family (namely, two elements a, a' in $\mathcal{A}$ give rise to the same element in $\widehat{\mathcal{A}}$ if a is obtained from a' by permuting the labels). Writing $a_n = \{a \in \mathcal{A} : |a| = n\}$ and $\widehat{a}_n = \{\widehat{a} \in \widehat{\mathcal{A}} : |\widehat{a}| = n\}$, prove that

$$\widehat{a}_n \leq a_n \leq \widehat{a}_n \cdot n!$$

`Problem 2 [20 points]:` *The Dyssimmetry Theorem for trees:* In the lectures we have proved that the generating function for labelled *rooted* trees $T(x)$ satisfies the equation $T(x) = xe^{T(x)}$. We have also seen that the GF associated to *unrooted* trees is equal to $T(x) - \frac{1}{2}T(x)^2$. Such an easy formula for a generating function suggests some direct explanation. This is what we will show in this problem. We start with the following easy observation:

- Prove that the center of a tree is a canonical root (namely, it is uniquely determined; the center however can be either a vertex or an edge).

If now $\mathcal{T}$ is a family of unrooted trees, write $\mathcal{T}_\bullet$ for the combinatorial family defined from $\mathcal{T}$ by marking one of the vertices. Similarly, define $\mathcal{T}_{\bullet-\bullet}$ and $\mathcal{T}_{\bullet\to\bullet}$, when marking and edge (and orienting it, in the second case).

- Prove the *Dyssimmetry Theorem for trees*: there is a combinatorial bijection between $\mathcal{T} \cup \mathcal{T}_{\bullet\to\bullet}$ and $\mathcal{T}_\bullet \cup \mathcal{T}_{\bullet-\bullet}$ (*Hint:* first, consider an element in $\mathcal{T}$ and root it canonically using its center, and see the position of the root with respect to the canonical root).
- Item apply the previous result to get $U(x)$ in terms of $T(x)$ without the integration step (namely, get $U(x) = T(x) - \frac{1}{2}T(x)^2$ using combinatorial arguments).
- `Bonus [10 points]` Application: use the result to get the generating function for unrooted unlabelled non-embedded trees, in terms of the generating function of rooted ones (*Hint*: here you should be careful with the family $\mathcal{T}_\bullet$: use the cycle-logarithmic operator).

`Problem 3 [10 points]:` *Unicyclic graphs:* An *unicyclic* graph is a graph (not necessarily connected) with exactly 1 cycle. Get the exponential generating function for this combinatorial family in terms of the generating function of rooted labelled trees $T(x)$ (which satisfies that $T(x) = x\exp(T(x))$) and the generating function $U(x)$ for unrooted labelled trees. In the generating function you obtain, use an auxiliar variable u in order to encode the length of this cycle (*Hint*: firstly, write the GF in terms of connected components. Later, see which is the structure of a connected unicyclic graph).

`Problem 4 [10 points]:` *Surjections and set partitions:* An *r-surjection* from $[n]$ to $[r]$ is a function $f : [n] \to [r]$ such that $f^{-1}(i) \neq \emptyset$ for all choices of i. An *r-partition* of $[n]$ is a partition of $[n]$ into r non-empty disjoint subsets.

1. Prove that the EGF of r-surjections, $S_r(x)$, where x marks the total number of preimages is equal to $(e^x - 1)^r$ (*Hint:* describe a surjection as a convenient finite sequence).
2. Prove that the EGF of r-partitions, $P_r(x)$, where x marks the total number of elements, is equal to $\frac{1}{r!}(e^x - 1)^r$. Relate is coefficients with the ones counting surjections.
3. From point 1.- deduce that the number of r-surjections of $[n]$ is equal to $\sum_{j=0}^{r} \binom{r}{j}(-1)^j(r-j)^n$, which is the formula you obtain when applying inclusion-exclusion.