Problem Sheet 3

Unlabelled trees and paths in the plane

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Discrete Mathematics III, Summer 2014
Deadline: 19th May 2014 (Monday) by 14:00, at the end of the lecture.

`Problem 1 [10 points]:` *The path length of a rooted tree:* Assume that we are studying Catalan trees (namely, general rooted trees embedded in the plane). The *path length* of a tree is the sum of distances (in terms of edges) from the root to all the vertices in the tree. Let $C(x,u)$ denote the GF for Catalan trees, where x marks vertices and u marks the path length. Prove that

$$C(x,u) = \frac{x}{1 - C(xu,u)}.$$

In fact, a similar argument works to show that we have something similar when dealing with the area under a Dyck path.

`Problem 2 [10 points]:` *Rooted trees with one leg:* recall that the GF of rooted embedded trees satisfies the relation $C(x) = \frac{x}{1-C(x)}$ (where x marks vertices).

1. Show that $C^\bullet(x) = xC'(x) = \frac{x}{1-2C(x)} = \frac{C(x)}{1-\frac{C(x)^2}{x}}$.
2. Give a combinatorial interpretation of the previous formula in terms of the unique path in a tree whose endvertices are the root of the tree and the marked vertex in the pointing operator.

`Problem 3 [20 points]:` *A continued fraction expansion for Dyck paths:*

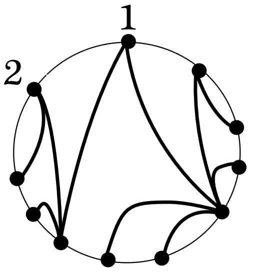
1. Show that the generating function for Dyck paths, where x counts the number of steps, can we written in the form
$$\cfrac{1}{1-\cfrac{x^2}{1-\cfrac{x^2}{1-\cfrac{x^2}{\ddots}}}}$$
2. Use these ideas in order to get the generating function of Dyck paths whose height is bounded by a certain constant l. Show that for every constant l, this GF is rational.
3. Using an auxiliar parameter u, encode into this GF the number of times that the Dyck path goes between a level $y = k$ and a level $y = k+1$ (or between level $y = k+1$ and level $y = k$: you count both ascending and descending steps).

`Problem 4 [10 points]:` *Motzkin paths:* A Motzkin path is a path similar to a Dyck path but with the extra step $(1,0)$ (in this case, a path along the axis $y = 0$ is possible). Using the the arguments used in Dyck paths, show that the GF for Motzkin paths satisfies the equation

$$M(x) = 1 + xM(x) + x^2M(x)^2$$

Writing $xM(x) = U(x)$, relate $[x^n]M(x)$ with Motzkin numbers. Finally, refine the previous equation by using an auxiliar parameter u in order to encode the number of steps of type $(1,0)$ (*Hint:* to get the equation, you would like to exploit the first step in the path in order to get a combinatorial decomposition).

`Problem 5 [20 points]:` *Non-crossing trees on a polygon:* consider n vertices on a circle, labelled with labels $[n]$. A non-crossing tree using these vertices is a tree whose vertex set is the n vertices such that there do not exist a pair of edges of the form $\overline{ac}$ and $\overline{bd}$, with $a < b < c < d$. Let $T(x)$ be the generating function of such trees, where x marks vertices.



1. A *butterfly* is an ordered pair of trees with a common root. Denote by $B(x)$ the generating function of butterflies, where x marks vertices. Show that

$$B(x) = \frac{1}{x}T(x)^2,\ T(x) = \frac{x}{1 - B(x)}.$$

   Deduce that $T(x)^3 - xT(x) + x^2 = 0$.

2. Write $T(x) = x + xy(x)$. Deduce that $y(x) = x(1 + y(x))^3$.
3. Apply Lagrange inversion formula to prove that $[x^n]T(x) = \frac{1}{2n-1}\binom{3n-3}{n-1}$.
4. Consider now the family of non-crossing *forests* on a polygon (now our graph has not cycles, but can be disconnected), and denote by $F(x, u)$ is GF, where u additionally encodes connected components. Prove that $F(x) = 1 + uT(xF(x))$.

(An analysis of this last equation can be also done by means of elimination theory of algebraic equations).