

Problem Sheet 2

The symbolic method: the basics

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Discrete Mathematics III, Summer 2014

Deadline: 12th May 2014 (Monday) by 14:00, at the end of the lecture.

Problem 1 [10 points]: *Some special type of compositions.* Get the OGF for

1. Integer compositions into k parts ($k \geq 3$), $x_1 + \dots + x_k = n$, where $1 \leq x_1 \leq 10$, $1 \leq x_2 \leq 5$ and x_3 is even.
2. Integer compositions into k parts, $x_1 + \dots + x_k = n$, where $x_1 < \dots < x_k$. Can you say something when we do not restrict the number of parts? (*Hint*: you should parametrize the problem using x_1 and the differences $x_2 - x_1, \dots, x_k - x_{k-1}$).

Problem 2 [10 points]: *Some special type of partitions.* Get the OGF for

1. Partitions of integers only using odd numbers.
2. Partitions of integers without repeated elements.

Can you find a relation between these two OGF? (In fact you should show that they are the same: the first bijective proof of this fact is due to Sylvester).

Problem 3 [10 points]: *A recurrence relation for integer partitions:* in the lectures we have proved that the GF for integer partitions is

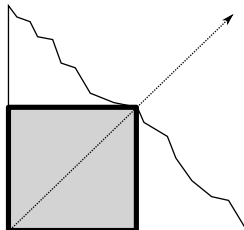
$$P(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{n=0}^{\infty} p_n x^n$$

Consider the derivative of the logarithm of this expression in order to show that

$$x \frac{P'(x)}{P(x)} = \sum_{n=1}^{\infty} \frac{nx^n}{1-x^n}$$

Using this expression, deduce the recurrence relation $p_n = \frac{1}{n} \sum_{j=1}^n \sigma(j) p_{n-j}$, where $\sigma(j)$ is the sum of the divisors of j (for instance, if $j = 6$, then $\sigma(j) = 1 + 2 + 3 + 6 = 12$). (*Hint*: once proved the first formula, try to give some interpretation to $\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n}$ in terms of σ).

Problem 4 [10 points]: *The Durfee square of partitions:* The diagram of any integer partition contains a uniquely determined square (known as the *Durfee square*) that is maximal, as exemplified by the following diagram



Using this decomposition, prove that

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{h \geq 0} \frac{x^{h^2}}{((1-x)(1-x^2)\dots(1-x^h))^2}.$$