Problem Sheet 1

Generating functions and its hierarchy

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Discrete Mathematics III, Summer 2014
Deadline: 5th May 2014 (Monday) by 14:00, at the end of the lecture.

`Problem 0 [0 points]:` (**Mandatory**) When presenting the solutions of the first problem sheet, you should write also:

- Your complete name and matriculation number.
- An e-mail which I can use to contact you.

`Problem 1 [20 points]:` Prove the following statements:

1. The formal power series $A(x)$ has functional inverse $B(x)$ if and only if $[x^0]A(x) = 0$, $[x^1]A(x) \neq 0$. Additionally, under these conditions, if the functional inverse exists, then it is unique.
2. Prove the chain rule for the derivative of a composition of formal power series (*Hint:* You may assume that you know which is the derivative of $A(x)^n$, which can be proved applying n times the derivative of the product).
3. For all positive integers p, the multiplicative inverse of the formal power series $(1-x)^{p+1}$ is $\sum_{n\geq 0}\binom{p+n}{n}x^n$.
4. If $A(x)$ is a Laurent series of order m, then $\mathrm{Res}\left(\frac{A'(x)}{A(x)}\right) = m$.

Comment: you can consider to choose any pair of these statements and prove them. This would be graded with at most 10 points (namely, this would be considered as one of the two problems). Then you should do another problem of the list.

`Problem 2 [10 points]:` we define the following formal power series on $\mathbb{C}[[x]]$:

$$S(x) := \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!}x^{2n+1},\ C(x) := \sum_{n\geq 0} \frac{(-1)^n}{(2n)!}x^{2n}$$

Using formal arguments over formal power series, show that:

1. $\frac{d}{dx}S(x) = C(x)$ and $\frac{d}{dx}C(x) = -S(x)$.
2. $S(x)^2 + C(x)^2 = 1$ (*Hint*: use the previous computation).

`Problem 3 [10 points]:` consider the following formal power series on $\mathbb{C}[[x]]$:

$$E(x) := \sum_{n\geq 0} \frac{1}{n!}x^n,\ L(x) := -\sum_{n\geq 1} \frac{1}{n}x^n$$

1. Show that $E(L(x)) = 1 - x$.
2. Using only formal arguments (no complex analysis...) show that $E(x)$ is **NOT** an algebraic formal power series on $\mathbb{C}[[x]]$ (*Hint:* apply the definition and try to get a contradiction).

`Problem 4 [20 points]:` let $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{m\geq 0} b_m x^m$ be formal power series on $\mathbb{C}[[x]]$. The *Hadamard product* of $A(x)$ and $B(x)$ is the formal power series $\sum_{n\geq 0} a_n b_n x^n$.

1. Prove that if the generating functions $A(x)$ and $B(x)$ are rational, then the Hadamard product is also rational.
2. Exhibit an example showing that the converse is **NOT** true.
3. Prove that if $A(x)$ is rational and $B(x)$ is algebraic, then the Hadamard product is algebraic (*Hint:* you should use that the derivative of an algebraic formal power seris is also algebraic)

In fact, the Hamadard product of two algebraic functions is **NOT** algebraic in general: the classical example is Catalan numbers (however, it is not obvious to prove that the sequence $\frac{1}{(n+1)^2}\binom{2n}{n}^2$ is not associated to an algebraic GF).

`Problem 5 [10 points]:` *Lagrange inversion formula, v2.* Following the lines of the proof done in the lecture, proof the following: let $U(x)$ be a formal power series satisfying that $U(x) = x\phi(U(x))$, for a certain $\phi(t)$ such that $[t^0]\phi(t) \neq 0$. Then,

$$[x^n]U(x)^k = \frac{k}{n}[t^{n-k}]\phi(t)^n.$$