The Dyssimmetry Theorem for trees

In the lectures we have proved that the generating function for labelled *rooted* trees $T(x)$ satisfies the equation $T(x) = xe^{T(x)}$. We have also seen that the GF associated to *unrooted* trees is equal to $T(x) - \frac{1}{2}T(x)^2$. Such an easy formula for a generating function suggests some direct explanation. This is what we will show in this problem.

We start with the following easy observation:

Claim: the center of a tree is a canonical root (namely, it is uniquely determined; the center however can be either a vertex or an edge). The proof of this fact is based in the following observation: two longest paths in a tree must intersect exactly once (if they do not intersect we would be able to construct a new path which is even longer; if they do intersect more than once we are creating a cycle). Hence, the intersection of all longest paths in a tree defines a unique structure (vertex or edge, depending on the parity of the length of the longest paths). This finally corresponds with the center of the tree.

If now $\mathcal{U}$ is a family of unrooted trees, write $\mathcal{T}_\bullet$ for the combinatorial family defined from $\mathcal{U}$ by marking one of the vertices. Similarly, define $\mathcal{T}_{\bullet-\bullet}$ and $\mathcal{T}_{\bullet\to\bullet}$, when marking and edge (and orienting it, in the second case). The *Dyssimmetry Theorem for trees* states that there is a combinatorial bijection between $\mathcal{U} \cup \mathcal{T}_{\bullet\to\bullet}$ and $\mathcal{T}_\bullet \cup \mathcal{T}_{\bullet-\bullet}$. Here combinatorial bijection means that the number of elements of a fixed size is the same in both sets. Let us prove it now.

We start splitting each of the sets in several parts, and we will show that we can define combinatorial bijections between them. Write:

- $\mathcal{U} = \mathcal{U}_v \cup \mathcal{U}_e$, where $\mathcal{U}_v$ is the set of trees in $\mathcal{U}$ whose center is a vertex, and $\mathcal{U}_e$ the set of trees in $\mathcal{U}$ whose center is an edge.
- $\mathcal{T}_\bullet = \mathcal{T}_{\bullet,c} \cup \mathcal{T}_{\bullet,1} \cup \mathcal{T}_{\bullet,2}$, where $\mathcal{T}_{\bullet,c}$ is the family of trees in $\mathcal{T}_\bullet$ whose center is the root vertex, $\mathcal{T}_{\bullet,1}$ is the family of trees in $\mathcal{T}_\bullet$ whose center is an edge which is incident with the root vertex and $\mathcal{T}_{\bullet,2}$ is the set containg the rest of the trees in $\mathcal{T}_\bullet$.
- $\mathcal{T}_{\bullet-\bullet} = \mathcal{T}_{\bullet-\bullet,c} \cup \mathcal{T}_{\bullet-\bullet,1}$, where $\mathcal{T}_{\bullet-\bullet,c}$ is the family of trees in $\mathcal{T}_{\bullet-\bullet}$ whose center is the root edge, and $\mathcal{T}_{\bullet-\bullet,1}$ are the rest of the trees.
- $\mathcal{T}_{\bullet\to\bullet} = \mathcal{T}_{\bullet\to\bullet,c} \cup \mathcal{T}_{\bullet\to\bullet,1} \cup \mathcal{T}_{\bullet\to\bullet,2}$, where $\mathcal{T}_{\bullet\to\bullet,c}$ is the family of trees in $\mathcal{T}_{\bullet\to\bullet}$ where the rooted edge is the center, $\mathcal{T}_{\bullet\to\bullet,1}$ is the family of trees where the root edge is oriented in the direction of the center, and $\mathcal{T}_{\bullet\to\bullet,2}$ is the family of trees where the root edge is oriented in the opposite direction.

In particular, in the last construction, observe that there is a unique path from the root to the center, hence the oriented edge can be oriented in the direction of the center or not. Putting all of this together, we would like to show that

$$\mathcal{U}_v \cup \mathcal{U}_e \cup \mathcal{T}_{\bullet\to\bullet,c} \cup \mathcal{T}_{\bullet\to\bullet,1} \cup \mathcal{T}_{\bullet\to\bullet,2} \simeq \mathcal{T}_{\bullet-\bullet,c} \cup \mathcal{T}_{\bullet-\bullet,1} \cup \mathcal{T}_{\bullet,c} \cup \mathcal{T}_{\bullet,1} \cup \mathcal{T}_{\bullet,2}.$$

So we will show that some terms on the left are combinatorially equal to some terms of the right. In particular:

1. $\mathcal{U}_v \cup \mathcal{U}_e \simeq \mathcal{T}_{\bullet,c} \cup \mathcal{T}_{\bullet-\bullet,c}$. This is obvious: the combinatorial bijection arises just noticing that the root and the center in this situation are the same (either a vertex or an edge).
2. $\mathcal{T}_{\bullet\to\bullet,c} \simeq \mathcal{T}_{\bullet,1}$: the bijection is done in the following way: once we have an element $t \in \mathcal{T}_{\bullet\to\bullet,c}$ we construct an element in $\mathcal{T}_{\bullet,1}$ by rooting the starting point of the root edge, and forgetting about the orientation of the root. This operation is clearly invertible!
3. $\mathcal{T}_{\bullet\to\bullet,1} \simeq \mathcal{T}_{\bullet,2}$: we apply a similar argument as in the previous case. Take a tree $t \in \mathcal{T}_{\bullet\to\bullet,1}$. By definition, the root edge is different of the center. Now we take the root vertex as the starting vertex in the oriented edge. In particular, this vertex is *not* incident with the center. This operation is clearly invertible again!

4. $\mathcal{T}_{\bullet\to\bullet,2} \simeq \mathcal{T}_{\bullet-\bullet,1}$: finally, having a tree $t \in \mathcal{T}_{\bullet-\bullet,1}$ we construct a tree in $\mathcal{T}_{\bullet\to\bullet,2}$ by orienting the root edge in the opposite direction in the path linking the root edge with the center. Again, this operation is trivially invertible.

Let us apply now this result in the context of rooted labelled trees. In this situation we have that the Dyssimmetry Theorem applies in the following form:

$$\mathcal{U} \cup \mathcal{T}_{\bullet\to\bullet} \simeq \mathcal{T}_{\bullet} \cup \mathcal{T}_{\bullet-\bullet},$$

and we want to express the GF of $\mathcal{U}$ in terms of the others. We have the following:

1. $T_{\bullet}(x)$ is the GF of trees which are rooted at a vertex. Hence this is equal to $T(x)$ satisfying that $T(x) = xe^{T(x)}$.
2. $\mathcal{T}_{\bullet\to\bullet}$: the root and oriented edge defines an ordered pair rooted labelled trees (pasted in the endvertices of the edge root). Hence we have that $\mathcal{T}_{\bullet\to\bullet} = \mathcal{T}_{\bullet} \times \mathcal{T}_{\bullet}$, and the corresponding GF is $T(x)^2$.
3. $\mathcal{T}_{\bullet-\bullet}$: here the construction is slightly different compared with $\mathcal{T}_{\bullet-\bullet}$: here the order does *not* matter, because the root edge is not oriented. hence, we have that $T_{\bullet-\bullet}(x) = T(x)^2/2$.

Combining now all this expressions we get that:

$$U(x) = T(x) + \frac{1}{2}T(x)^2 - T(x)^2 = T(x) - \frac{1}{2}T(x)^2,$$

which is the result we obtained by direct integration.

As an extra application of the method, let us find the GF for unrooted unlabelled non-embedded trees, in terms of the generating function of rooted ones. In this situation the single change arises in the following two facts:

1. Instead of having the base GF $T(x) = xe^(T(x))$ now we have a GF $t(x)$ satisfying that $t(x) = x \exp\left(\sum_{r\geq 1} \frac{1}{r} t(x^r)\right)$.
2. The GF associated to $\mathcal{T}_{\bullet-\bullet}$ is now $\frac{T(x)^2+T(x^2)}{2}$: as the trees now they do not have labels, a permutation of the two rooted subtrees would give the same final tree.

In this final situation we would have that

$$u(x) = t(x) - \frac{t(x)^2}{2} - \frac{t(x^2)}{2}.$$