Let's go to study some examples using this language:

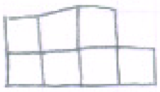
Ex/ Let P be the set of words over the alfabet $\{0,1\}$. Then $P = Seq(A)$, where $A = \{0,1\}$. As the size of a word is its lenght, we have that $A(x) = 2x$, hence

$$P(x) = (1-2x)^{-1} \rightsquigarrow [x^n]\, P(x) = 2^n$$

Ex/ Compositions and partitions: given an integer n, a composition of n is a solution of the equation $x_1 + \ldots + x_k = n$, where $1 \le x_i \in \mathbb{N}$, and k is arbitrary (composition into k-parts). Hence, $\mathcal{C} = Seq(\mathbb{N}_{\ge 0})$, where $\mathbb{N}_{\ge 0} = \{1, 2, \ldots\}$; and the size of each element is its value. Then,

$$C(x) = \Big(1 - \sum_{i=1}^{\infty} x^i\Big)^{-1} = \frac{1}{1 - \frac{x}{1-x}} = \frac{1-x}{1-2x} \rightarrow 2^{n-1}$$

$$C(x,u) = \Big(1 - u \sum_{i=1}^{\infty} x^i\Big)^{-1} = \frac{1}{1 - u\frac{x}{1-x}} = \frac{1-x}{1-(1+u)x} \rightarrow (1+u)^n - (1+u)^{n-1}$$


Young tableaux

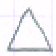
Partitions are defined similarly as compositions, but now the order does NOT matter. Hence,

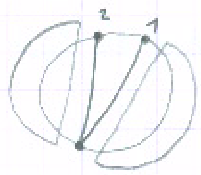
$$\mathcal{P} = Seq_{\ge 0}\{1\} \times Seq_{\ge 0}\{2\} \times \ldots \Rightarrow P(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i} \rightsquigarrow [x^n]\, P(x) \sim \frac{1}{4n\sqrt{3}} \exp\Big(\pi\sqrt{\tfrac{2n}{3}}\Big)$$
$$= Mul(\mathbb{N}_{\ge 0})$$

If we restrict ourselves to a set of integers $\Delta \subseteq \mathbb{N}$, then we have $\prod_{i \in \Delta} \frac{1}{1-x^i}$.

Ex/ Triangulations of a polygon: let $\mathcal{C}$ be the class of triangulations of a labelled polygon. Then, in this situation the size of an element is the number of triangles

$|e| = 0$


$|\Delta| = 1$

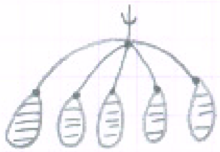


$$\mathcal{C} = e \cup \mathcal{C} \times \Delta \times \mathcal{C}$$
$$\Downarrow$$
$$C(x) = 1 + x\,C(x)^2 \Rightarrow C_n = \frac{1}{n+1}\binom{2n}{n}$$


root of the tree ≡ leg

Ex/ Rooted trees. We consider now two distinct families of trees which are counted by Catalan numbers.

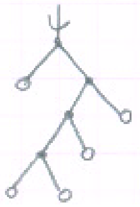
a) General trees. In this case there is no obstruction on the vertex degree:



Embedded $\Rightarrow \mathcal{T} = \circ \times Seq_{\ge 0}(\mathcal{T}) \Rightarrow T(x) = \frac{x}{1-T(x)} \rightarrow T(x) = x + T(x)^2$

Non-embedded $\Rightarrow \mathcal{T} = \circ \times MSet(\mathcal{T}) \Rightarrow T(x) = x \exp\Big(\sum_{r=1}^{\infty} \frac{1}{r} T(x^r)\Big)$

b) Binary trees: a binary tree is a tree where all internal vertices have degree 3 (the root, joint with the leg, have degree 3). Here the enumeration is done with respect to the number of internal vertices.



Embedded $\Rightarrow \mathcal{B} = \circ \cup \mathcal{B} \times \bullet \times \mathcal{B} \Rightarrow B(x) = 1 + xB(x)^2 \rightarrow B_n = \frac{1}{n+1}\binom{2n}{n}$

Non-embedded $\Rightarrow \mathcal{B} = \circ \cup MSet_2(\mathcal{B}) \Rightarrow B(x) = 1 + \frac{x}{2}\big(B(x)^2 + B(x^2)\big)$