

The root of a map

Remember that a map is an embedding of a connected planar multigraph in the sphere such that edges just intersect in the vertices. The aim of this short note is to clarify a little more the notion of root in a map.

Maps are unlabeled objects. Hence, they are subject to symmetries, which makes a combinatorial treatment more complicated. Precisely, given two maps M_1 and M_2 , an *isomorphism* from M_1 to M_2 is an orientation preserving homeomorphism f of the sphere such that sends the associated multigraph of M_1 to the one of M_2 . To avoid problems due to automorphisms of the map (namely, non trivial homomorphisms which transform the associated multigraph, but which leaves the multigraph equal, due to the fact that this object is unlabelled), it is more convenient to consider *rooted* maps instead of maps. A map is said to be *rooted* by marking and orienting an edge so that this edge has the outer face on its left. The marked oriented edge is called the *root* and its origin is called the *root-vertex*. If no outer face is specified (map on the sphere rather than on the plane), the operation of rooting consists in marking an edge and choosing the face on the left as outer-face, so as to get a rooted planar map. Saying which face is the rooted one is equivalent to fix the orientation of the sphere.

In this context, isomorphisms between rooted maps needs also the additional condition that the roots of the two corresponding two maps are also mapped. It is not difficult then to prove that a rooted map has no symmetry (namely, that the automorphism group of the rooted map is trivial).

Notice that this property is not true for unlabeled planar graphs, i.e., a planar graph with a marked oriented edge can have nontrivial automorphisms. This is in fact true due to the embedding in the sphere, which enriches the structure under study.

Rooted maps can also be defined by pointing one of the corners: the root face is the one containing the corner, and the root vertex is the vertex pointed by the corner. Finally (and equivalently to choose and orientation of the sphere) the root edge will be the first edge we find in a contour walk on the root face, starting at the root vertex in anti clockwise order.

Just to point out two examples: here we draw two maps with the root-vertex, the root-edge and the root-face. The root corner (in green) and the root edge (parallel to the real root edge) are also indicated.

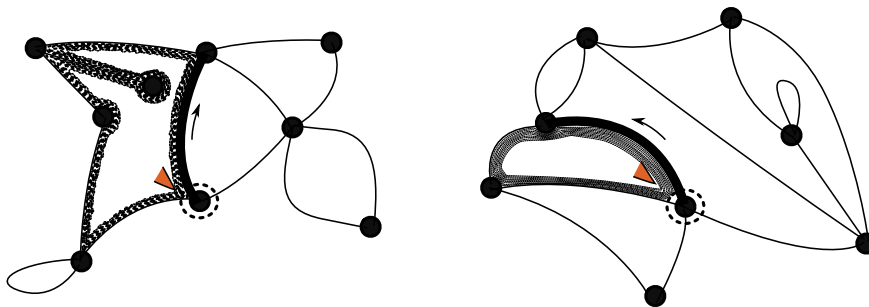


FIGURE 1. Two examples of maps.