The transfer theorem for singularity analysis and the Lagrangian Scheme

In this note we state the results claimed in the lecture concerning the subexponential growth of coefficients. In particular, we will cover the results concerning the Lagrangian Scheme. In this situation, assume that $A(z)$ is a function satisfying that $A(z) = z\phi(A(z))$, where the function $\phi(u)$ satisfies the following properties:

(H1) $\phi(u)$ is an analytic function at $u = 0$, $\phi(0) \neq 0$, $[u^n]\phi(u) \geq 0$, $\phi(u) \neq \phi_0 + \phi_1 u$ (ϕ is not a linear function).

(H2) If R is the radius of convergence of ϕ, then there is a unique τ, $0 < \tau < R$ such that $\phi(\tau) = \tau\phi'(\tau)$.

Under these technical conditions we can assure that the combinatorial problem is well defined, and we will be able to get general results for asymptotic estimates. In particular, we know that $\psi(u) = u/\phi(u)$ is the inverse function of $A(z)$, which means that $A(z)$ ceases to be analytic at the point ρ where $\psi'(A(\rho)) = 0$. Writing $A(\rho) = \tau$ we have that:

$$\psi'(\tau) = 0 \to \left(\frac{\tau}{\phi(\tau)}\right)' = 0 \to \frac{\phi(\tau) - \tau\phi'(\tau)}{\phi(\tau)^2} = 0,$$

which gives precisely the condition $(H2)$ in the Lagrangian scheme setting. This means that $\rho = \psi(\tau) = \tau/\phi(\tau)$ is precisely the singularity of $A(z)$.

Now that we know where the singularity is, we should see which is its nature, and this will give us the subexponential growth of the coefficients. As the singularity arises from the fact that ψ' vanishes, we would wonder which is the expected behaviour of $A(z)$ around ρ. This can be argued by taking the Taylor expansion of ψ around τ.

$$\psi(u) \simeq \psi(\tau) + \frac{\psi''(\tau)}{2}(u - \tau)^2 + ...,$$

where we are assuming that $\psi''(\tau) \neq 0$ (which is what usually happens in combinatorial applications). Then we can try to invert this relation by writing $u = A(z)$. We then get that

$$A(z) = A(\rho) \pm \sqrt{\frac{2}{\psi''(\tau)}}(z - \rho)^{1/2} + ...$$

where we need to decide which sign we choose. These type of expansions (similar to Taylor series, but where the exponents can be fractions) are called *singular expansions*, and generalize what happen in the context of algebraic functions (where we have the so-called Puiseux expansions).

The important thing here is that we would like to extract asymptotic estimates from a singular expansion. In particular, we would like to say something like this: imagine that $A(z)$ has a singular expansion of the form

$$A(z) = a_0 + a_1(1 - z/\rho)^{-\alpha} + O\left((1 - z/\rho)^{-\alpha+1}\right).$$

So we would like to say that

$$[z^n]A(z) = [z^n]a_0 + a_1[z^n](1 - z/\rho)^{-\alpha} + O\left([z^n](1 - z/\rho)^{-\alpha+1}\right).$$

This is far of being obvious but in fact it is true. This is what is called the *Transfer Theorem in Analytic Combinatorics*, and was firstly proved by Flajolet and Odlyzko in the 80's:

Theorem 1 (Transfer Theorem for singularity analysis, Flajolet & Odlyzko) *If $A(z)$ has its smallest singularity at $z = \rho$, and there it has an expansion of the form*

$$A(z) = a_0 + a_1\left(1 - z/\rho\right)^{-\alpha} + O\left(\left(1 - z/\rho\right)^{-\alpha+1}\right)$$

Then, if $-\alpha$ is not a positive integer we have that

$$[z^n]A(z) = a_1\frac{n^{\alpha-1}}{\Gamma(\alpha)}\rho^{-n}(1 + o(1)).$$

So, how do we get the term $\frac{n^{\alpha-1}}{\Gamma(\alpha)}$? Well, we know that if $-\alpha$ is not a positive integer, then by Newton's formula for the binomial:

$$[z^n](1-z/\rho)^{-\alpha} = \binom{-\alpha}{n}(-\rho)^{-n} = \frac{(-\alpha)(-\alpha-1)\dots(-\alpha-n+1)}{n!}(-\rho)^{-n} = \frac{(\alpha)(\alpha+1)\dots(\alpha+n-1)}{n!}\rho^{-n}.$$

Now this term can be written as

$$\frac{(n+\alpha-1)\dots(\alpha+1)\alpha}{n!}\rho^{-n} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)},$$

where $\Gamma(\alpha)$ is the integral $\int_0^\infty t^{\alpha-1}e^{-t}dt$. Now using Stirling's approximation formula:

$$\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} = \frac{\sqrt{2\pi(n+\alpha-1)}\left(\frac{n+\alpha-1}{e}\right)^{n+\alpha-1}}{\Gamma(\alpha)\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}(1+o(1)) \simeq \frac{n^{\alpha-1}}{\Gamma(\alpha)}(1+o(1)),$$

where we have repeatedly used that α is small compared with n (n is VERY big). In particular, you have to be very accurate on the approximations: for instance, you would like to write:

$$\frac{(n+\alpha-1)^{n+\alpha-1}}{n^n} \simeq \left(1+\frac{\alpha-1}{n}\right)^n n^{n+\alpha-1} \simeq e^{\alpha-1}n^{\alpha-1},$$

and the exponential in e cancels with the rest of the exponentials in the long expression.

Going back to Lagrangian schemes, this tells us that the asymptotic estimate in this setting will be always of the form:

$$a_1\frac{n^{-1/2-1}}{\Gamma(-1/2)}\rho^{-n}(1+o(1)) = a_1\frac{n^{-3/2}}{-2\sqrt{\pi}}\rho^{-n}(1+o(1)).$$

Hence, we need to take the negative sign in the singular expansion.

Let us do now a complete example to show this methodology. Consider the Catalan function $C(z)$ which satisfies that $C(z) = 1+zC(z)^2$. For convenience, write $U(z) = C(z)-1$, obtaining that $U(z) = z(1+U(z))^2$. Hence, the singularities of U and C are the same (they differ of just 1, which is an analytic function!), and consequently $[z^n]U(z) = [z^n]C(z)$. Here, $\phi(u) = (1+u)^2$. Observe that condition $(H1)$ is trivially satisfied. Concerning $(H2)$, the radius of convergence here is $R=\infty$, and the solution to the equation is

$$2\tau(1+\tau) = (1+\tau)^2 \to 2\tau = (1+\tau) \to \tau = 1.$$

because τ cannot be -1. Then, the singularity is located at $1/\phi(1) = 1/4$. Now we know that around this point, the function $U(z)$ admits a singular expansion of the form:

$$U(z) = u_0 + u_1X + u_2X^2 + u_3X^3 + \dots$$

where we write $X = (1-x/\rho)^{1/2} = (1-4x)^{1/2}$ (and in particular, $z = \frac{1}{4}(1-X^2)$). Now we want to get the coefficients u_0 and u_1 in order to apply the Transfer Theorem. To do so, wa write the singular expansion in the equation for U, and see that term by term we have cancelations. More precisely:

$$U(z) = z(1+U(z))^2 \to u_0+u_1X+u_2X^2+u_3X^3+\dots = \rho(1-X^2)(1+u_0+u_1X+u_2X^2+u_3X^3+\dots)^2.$$

The term of X^0 in this equality is the relation $u_0 = \rho(1+u_0)^2$, which gives us that u_0 is equal to 1. In fact, u_0 is $U(\rho)$, which is equal to the value of τ.

Concerning the term X^1, we get the relation $u_1 = \rho(2u_1(1+u_0))$, which gives $u_1 = u_1$. The second equation does not give any information about u_1. This is due to the cancelation of the first derivative. Finally, the term of X^2 gives that $u_2 = \rho(u_1^2 + 2(1+u_0)u_2 - (1+u_0)^2)$. In this situation there is a magical cancelation of u_2 (due to the cancelation of the first derivative of the inverse function of $U(z)$), and we conclude that $u_1 = \pm 2$.

From these two values we need to take the one that gives positive coefficients. When this is the factor of the square root, this is always the term with $-$. Hence, the singular expansion of $U(z)$ at $z = 1/4$ is equal to:

$$U(z) = 1 - 2(1-4z)^{1/2} + \dots$$

Now we can apply the transfer theorem, which states that:

$$[z^n]U(z) = -2\frac{n^{-3/2}}{\Gamma(-1/2)}4^n(1+o(1)) = -2\frac{n^{-3/2}}{-2\sqrt{\pi}}4^n(1+o(1)) = \frac{n^{-3/2}}{\sqrt{\pi}}4^n(1+o(1))$$

which is precisely the estimate for Catalan numbers we had by applying directly Stirling formula (notice here the necessity of using the negative value in order to get a positive estimate).