

## Sum of a random number of random variables

October 4, 2013

1 / 14

## Contents

Sum of a random number of random variables

Examples

Expected values

2 / 14

## Sum of a random number of random variables

Let  $X_1, X_2, \dots, X_n, \dots$  and  $N$  be jointly independent random variables such that

- ▶  $X_1, X_2, \dots, X_n, \dots$  are identically distributed.
- ▶  $N$  is a discrete random variable taking the values  $k = 0, 1, 2, 3, \dots$

Let us consider the sum

$$S = X_1 + X_2 + \dots + X_N$$

(Let  $S = 0$  if  $\{N = 0\}$  happens.)

3 / 14

## Sum of a random number of random variables

The characteristic function of  $S$  can be obtained as follows:

$$M_S(\omega) = E\left(e^{i\omega S}\right) = \sum_{k \geq 0} E\left(e^{i\omega S} \mid N = k\right) P(N = k)$$

We have  $E\left(e^{i\omega S} \mid N = 0\right) = E\left(e^{i\omega 0}\right) = 1$ . Moreover, for  $k \geq 1$ ,

$$\begin{aligned} E\left(e^{i\omega S} \mid N = k\right) &= E\left(e^{i\omega(X_1 + X_2 + \dots + X_N)} \mid N = k\right) \\ &= E\left(e^{i\omega(X_1 + X_2 + \dots + X_k)} \mid N = k\right) = E\left(e^{i\omega(X_1 + X_2 + \dots + X_k)}\right) \\ &= E\left(e^{i\omega X_1}\right) E\left(e^{i\omega X_2}\right) \dots E\left(e^{i\omega X_k}\right) = (M_X(\omega))^k \end{aligned}$$

4 / 14

## Sum of a random number of random variables

Let

$$G_N(z) = \sum_{k \geq 0} z^k P(N = k)$$

be the probability generating function of  $N$ .

Then,

$$M_S(\omega) = \sum_{k \geq 0} (M_X(\omega))^k P(N = k) = G_N(M_X(\omega))$$

- ▶ Notice that  $|M_X(\omega)| \leq 1$  for all  $\omega \in \mathbb{R}$  and that  $G_N(z)$  converges for all  $z \in \mathbb{C}$  such that  $|z| \leq 1$ .

5 / 14

## Example 1

- ▶ Suppose that the number  $N$  of costumers that arrive at a service point is a  $\text{Poiss}(\lambda)$  random variable.
- ▶ Let  $p$  be the probability that a customer who arrives at the service point will be actually served.

Let us calculate the probability distribution of the number  $S$  of served customers.

6 / 14

## Example 1

We have

$$S = X_1 + X_2 + \dots + X_N,$$

where  $X_i$  is the indicator of the event “the  $i$ -th arrival has been served”

Moreover,

$$M_{X_i}(\omega) = M_X(\omega) = pe^{i\omega} + q, \quad \omega \in \mathbb{R}$$

$$G_N(z) = e^{\lambda(z-1)}, \quad z \in \mathbb{C},$$

7 / 14

## Example 1

Therefore

$$\begin{aligned} M_S(\omega) &= G_N(M_X(\omega)) = e^{\lambda(z-1)} \Big|_{z=pe^{i\omega}+q} \\ &= e^{\lambda(pe^{i\omega}+q-1)} = e^{\lambda p(e^{i\omega}-1)} \end{aligned}$$

This is the characteristic function of a Poisson random variable with parameter  $\lambda p$ .

Thus

$$S \sim \text{Poiss}(\lambda p)$$

8 / 14

## Example 1

Analogously, if  $R$  denotes the number of non-served customers,

$$R \sim \text{Poiss}(\lambda q)$$

- ▶ It can be proved that  $S$  and  $R$  are independent random variables
- ▶ Notice how the convolution theorem applies. Indeed,  $S + R = N$  and, thus,

$$\begin{aligned} M_S(\omega) M_R(\omega) &= e^{\lambda p(e^{i\omega}-1)} e^{\lambda q(e^{i\omega}-1)} \\ &= e^{\lambda(p+q)(e^{i\omega}-1)} = e^{\lambda(e^{i\omega}-1)} = M_N(\omega) \end{aligned}$$

9 / 14

## Example 2

Suppose now that each  $X_i \sim \text{Exp}(\mu)$ , that  $N \sim \text{Geom}(p)$ , and let

$$S = X_1 + X_2 + \dots + X_N$$

Then

$$\begin{aligned} M_X(\omega) &= \frac{\mu}{\mu - i\omega}, \\ G_N(z) &= \frac{zp}{1 - zq} \end{aligned}$$

10 / 14

## Example 2

Therefore

$$\begin{aligned} M_S(\omega) &= G_N(M_X(\omega)) = \frac{zp}{1 - zq} \Big|_{z=\mu/(\mu-i\omega)} \\ &= \frac{\frac{\mu}{\mu-i\omega} p}{1 - \frac{\mu}{\mu-i\omega} q} = \frac{\mu p}{\mu p - i\omega} \end{aligned}$$

This is the characteristic function of a exponential r.v. with parameter  $\mu p$ .

Hence,

$$S \sim \text{Exp}(\mu p)$$

11 / 14

## Expected values

If

$$S = X_1 + X_2 + \dots + X_N,$$

we can obtain the expected value of  $S$  calculating first the conditional expected value given  $N$ .

$$E(S) = E(E(S | N))$$

12 / 14

## Expected values

But

$$\begin{aligned} E(S \mid N = k) &= E(X_1 + X_2 + \cdots + X_k \mid N = k) \\ &= E(X_1 + X_2 + \cdots + X_k) = m k, \end{aligned}$$

where  $m$  denotes the mean value of each  $X_i$ .

Therefore

$$E(S \mid N) = mN$$

and, so,

$$E(S) = E(E(S \mid N)) = E(mN) = mE(N) = E(N)E(X)$$

## Expected values

- ▶ In Example 1,  $X \sim B(p)$  and  $N \sim \text{Poiss}(\lambda)$ . Then  $S \sim \text{Poiss}(\lambda p)$ . Thus,

$$E(S) = \lambda p = E(N) E(X)$$

- ▶ In Example 2,  $X \sim \text{Exp}(\mu)$ ,  $N \sim \text{Geom}(p)$ . Then  $S \sim \text{Exp}(\mu p)$ . Hence

$$E(S) = \frac{1}{\mu p} = \frac{1}{p} \cdot \frac{1}{\mu} = E(N) E(X)$$