

Introduction to Stochastic Processes

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Stochastic processes

Stochastic (or random) process: a mathematical model of a magnitude that evolves as time passes in such a way that for any fixed instant of time one has a random variable.

Definition

Let (Ω, \mathcal{F}, P) be a probability space and let $R_t \subset \mathbb{R}$.

A stochastic process X is a mapping

$$X : R_t \times \Omega \longrightarrow \mathbb{R} \\ (t, \omega) \mapsto X(t, \omega)$$

such that $X(t, \omega)$ is a random variable for all $t \in R_t$.

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Stochastic processes

- ▶ Since $X(t, \omega)$ is a random variable for any fixed $t \in R_t$, we have that the subset of outcomes

$$\{\omega \in \Omega : X(t, \omega) \leq x\}$$

is an event (an element of \mathcal{F}) for all $x \in \mathbb{R}$.

- ▶ In many applications the parameter t corresponds to **time**.
- ▶ The dependence on ω is usually omitted and one writes X or $X(t)$ to denote the process.

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Stochastic processes

Therefore,

- ▶ A stochastic process X is a collection of random variables indexed by the parameter t (time):

$$\{X(t, \omega) \equiv X_t(\omega), t \in R_t\}$$

- ▶ X is also a collection of functions of time:

$$\{X(t, \omega) \equiv X_\omega(t), \omega \in \Omega\},$$

where $X_\omega(t)$ is the **realization** of the process corresponding to the outcome $\omega \in \Omega$.

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Discrete-time processes

When R_t is a discrete set (finite or denumerable) we say that X is a **discrete-time** process.

- ▶ If $R_t = \{t_1, t_2, \dots, t_n\}$ is finite, then

$$X = (X(t_1), X(t_2), \dots, X(t_n))$$

is a random vector.

- ▶ If $R_t = \{t_1, t_2, \dots, t_n, \dots\}$ is countable, then

$$X[n] = \{X(t_1), X(t_2), \dots, X(t_n) \dots\}$$

is a sequence of random variables.

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Continuous-time processes

When R_t is uncountable one says that X is a **continuous-time** process.

Usual cases are:

$$X(t), t \in [a, b]$$

$$X(t), t \in (0, \infty)$$

$$X(t), t \in \mathbb{R}$$

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State space

According to the values taken by the process we have the following classification:

- ▶ $X(t)$ is a **discrete-state** process if its values are countable, that is, if the image set of the mapping X is finite or denumerable.
- ▶ Otherwise, $X(t)$ is a **continuous-state** process.

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Distribution and density functions

Consider (for any fixed $t \in R_t$) the probability distribution function of the random variable $X(t)$:

$$F_X(x; t) \equiv F_{X(t)}(x) = P(X(t) \leq x)$$

- ▶ $F_X(x; t)$ is the **first-order distribution function** of the process.
- ▶ It describes the time evolution of the process probability law.

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Distribution and density functions

For a continuous-state process (such that the random variable $X(t)$ is continuous for any t) we can consider the **first-order density**:

$$f(x; t) \Delta x \approx P\{x \leq X(t) \leq x + \Delta x\}$$

More formally,

$$f(x; t) = \lim_{\Delta x \rightarrow 0} \frac{P\{x \leq X(t) \leq x + \Delta x\}}{\Delta x}$$

- ▶ $f_X(x; t)$ is for any $t \in R_t$ the probability density $f_{X(t)}(x)$ of the random variable $X(t)$.

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Distribution and density functions

The known relations between distribution and density functions hold. For instance,

$$f_{X(t)}(x) = \frac{d}{dx} F_{X(t)}(x),$$

i.e.,

$$f_X(x; t) = \frac{\partial F_X(x; t)}{\partial x}$$

Analogously,

$$F_{X(t)}(x) = \int_{-\infty}^x f_{X(t)}(u) du,$$

i.e.,

$$F_X(x; t) = \int_{-\infty}^x f_X(u; t) du$$

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Distribution and density functions

If $X(t)$ is a discrete-state process taking the values $x_1, x_2, \dots, x_k, \dots$, we can define the **first-order probability function**:

$$p_X(x_k; t) \equiv p_{X(t)}(x_k) = P(X(t) = x_k), \quad k \geq 1$$

We have that

$$F_X(x; t) = \sum_{x_k \leq x} p_X(x_k; t).$$

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Distribution and density functions

Definition

The n -order distribution function of the process $X(t)$ is the joint distribution function of the random variables

$$X(t_1), X(t_2), \dots, X(t_n)$$

arising from $X(t)$ at arbitrary instants of time $t_1, t_2, \dots, t_n \in R_t$:

$$\begin{aligned} F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) \\ &\equiv F_{X(t_1), X(t_2), \dots, X(t_n)}(x_1, x_2, \dots, x_n) \\ &= P(X(t_1) \leq x_1, X(t_2) \leq x_2, \dots, X(t_n) \leq x_n) \end{aligned}$$

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Distribution and density functions

For a continuous-state process the n -order density function is obtained as:

$$\begin{aligned} f_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) \\ = \frac{\partial^n F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n)}{\partial x_1 \partial x_2 \cdots \partial x_n} \end{aligned}$$

It is to the joint density of the random variables $X(t_1), X(t_2), \dots, X(t_n)$.

Analogously, if $X(t)$ is a discrete-state process, the n -order probability function is

$$\begin{aligned} p_X(x_{k_1}, x_{k_2}, \dots, x_{k_n}; t_1, t_2, \dots, t_n) \\ = P(X(t_1) = x_{k_1}, X(t_2) = x_{k_2}, \dots, X(t_n) = x_{k_n}) \end{aligned}$$

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Distribution and density functions

The usual relations and properties of these functions hold. For example,

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x; t) dx &= 1 \\ \int_{-\infty}^{\infty} f_X(x_1, x_2; t_1, t_2) dx_1 &= f_X(x_2; t_2) \\ \lim_{x_2 \rightarrow \infty} F_X(x_1, x_2; t_1, t_2) &= F_X(x_1; t_1) \\ p_X(x_{k_1}; t_1) &= \sum_{k_2} p_X(x_{k_1}, x_{k_2}; t_1, t_2) \end{aligned}$$

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Mean and autocorrelation

Definition

The mean function of the process is $m_X(t) = E(X(t))$

- ▶ For any $t \in R_t$, $m_X(t)$ is the expected value of the random variable $X(t)$.
- ▶ For a continuous-state process

$$m_X(t) = \int_{-\infty}^{\infty} x f_X(x; t) dx$$

- ▶ For a discrete-state process

$$m_X(t) = \sum_k x_k p_X(x; t)$$

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Mean and autocorrelation

For discrete-time processes the above formulas are

$$m_X[n] = E(X[n])$$

$$= \begin{cases} \int_{-\infty}^{\infty} x f_X(x; n) dx, & \text{continuous-state} \\ \sum_k x_k p_X(x_k; n), & \text{discrete-state} \end{cases}$$

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Mean and autocorrelation

Definition

The autocorrelation function of the process is defined as

$$R_X(t_1, t_2) = E(X(t_1)X(t_2)).$$

- ▶ $R_X(t_1, t_2)$ is the joint moment m_{11} of the random variables $X(t_1)$ and $X(t_2)$.
- ▶ It is a symmetric function, i.e. $R_X(t_1, t_2) = R_X(t_2, t_1)$.
- ▶ For $t_1 = t_2 = t$ we have that $R_X(t, t) = E(X^2(t))$. This function of time is called the **average power** of $X(t)$.

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Mean and autocorrelation

- ▶ If $X(t)$ is a continuous-state process,

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2) dx_1 dx_2,$$

- ▶ Otherwise, if $X(t)$ a discrete-state process we have that

$$R_X(t_1, t_2) = \sum_{k_1} \sum_{k_2} x_{k_1} x_{k_2} p_X(x_{k_1}, x_{k_2}; t_1, t_2).$$

- ▶ For discrete-time processes we can write analogous expressions.

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Autocovariance

Definition

The autocovariance function of the process is defined as

$$C_X(t_1, t_2) = E((X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))).$$

- ▶ Expanding this expression we obtain

$$C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1)m_X(t_2)$$

- ▶ Notice that

$$C_X(t, t) = E((X(t) - m_X(t))^2)$$

is the variance $\sigma_{X(t)}^2$ of the random variable $X(t)$.

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Stationary processes

$X(t)$ is a **strict-sense stationary** process if its statistical properties are invariant to a shift of the origin of time.

This means that the processes $X(t)$ and $X(t + c)$ have the same statistics for any $c > 0$.

Definition

The process $X(t)$ is strict-sense stationary if the families of random variables

$$\{X(t_1), X(t_2), \dots, X(t_n)\}$$

and

$$\{X(t_1 + c), X(t_2 + c), \dots, X(t_n + c)\}$$

have the same joint distribution for all t_1, t_2, \dots, t_n and $c > 0$.

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Stationary processes

Therefore, for all $n \geq 1$, for all $c > 0$ and for all t_1, t_2, \dots, t_n , we have that

$$\begin{aligned} F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) \\ = F_X(x_1, x_2, \dots, x_n; t_1 + c, t_2 + c, \dots, t_n + c). \end{aligned}$$

- ▶ When this condition is satisfied for $n = 1, 2, \dots, k$ we say that $X(t)$ is **k -order stationary**.
- ▶ Hence, a strict-sense stationary process is k -order stationary for any $k \geq 1$.

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Stationary processes

In particular, taking $n = 1$ we have that

$$F_X(x; t) = F_X(x; t + c) \quad \text{for all } c$$

That is, the first-order probability distribution function of a stationary process is time-invariant:

$$F_X(x; t) \equiv F_X(x)$$

Analogously, if $X(t)$ is a continuous-state process:

$$f_X(x; t) \equiv f_X(x)$$

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Stationary processes

Theorem

The mean function of a stationary stochastic process is constant.

Proof:

$$m_X(t) = E(X(t)) = \int_{-\infty}^{\infty} x f_X(x; t) dx = \int_{-\infty}^{\infty} x f_X(x) dx \equiv m_X.$$

- ▶ More generally, any property involving a single random variable arising from the process will be time-invariant.

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Stationary processes

Taking $n = 2$,

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; t_1 + c, t_2 + c) \quad \text{for all } c$$

That is,

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; 0, t_2 - t_1),$$

Therefore

$$F_X(x_1, x_2; t_1, t_2) \equiv F_X(x_1, x_2; \tau), \quad \tau = t_2 - t_1$$

- ▶ The joint distribution of two random variables $X(t_1)$, $X(t_2)$ arising from the process only depends on the lag $\tau = t_2 - t_1$ between the two instants t_1 and t_2 , but not on the absolute origin of time.

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Stationary processes

Theorem

The autocorrelation function of a k -order stationary process ($k \geq 2$) only depends on the lag τ :

$$R_X(\tau) = E(X(t)X(t + \tau)),$$

where this expectation is independent of t .

In particular, taking $t = 0$:

$$R_X(\tau) = E(X(0)X(\tau)).$$

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Properties of $R_X(\tau)$

- ▶ $R_X(\tau)$ is an even function:

$$R_X(-\tau) = E(X(t)X(t - \tau)) = E(X(t' + \tau)X(t')) = R_X(\tau)$$

Hence, we can take $\tau = |t_1 - t_2|$.

- ▶ The average power of $X(t)$ is constant:

$$E(X^2(t)) = R_X(0) \geq 0$$

- ▶ We have that $|R_X(\tau)| \leq R_X(0)$ for any τ .

Indeed,

$$0 \leq E((X(0) \pm X(\tau))^2) = 2R_X(0) \pm 2R_X(\tau)$$

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Wide-sense stationary processes

Definition

A stochastic process $X(t)$ is wide-sense stationary if

- ▶ The mean function is time-invariant:

$$m_X(t) = m_X.$$

- ▶ The autocorrelation depends only on τ :

$$R_X(t, t + \tau) = E(X(t)X(t + \tau)) = R_X(\tau).$$

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Wide-sense stationary processes

For time-discrete processes the above definition is translated as:

Definition

A random sequence $X[n]$ is wide-sense stationary if

- ▶ The mean is constant, that is $m_X[n] = m_X[0]$ for all n .
- ▶ The autocorrelation $R_X[n, n + k] \equiv R[k]$ depends only on k .