Probability and Random Processes

Exercises and Problems

Convergence of sequences of r.v.

1. Use the continuity theorem to prove that the distribution of $X_n \sim \mathsf{Bin}(n, \lambda/n)$ converges to a $\mathsf{Poiss}(\lambda)$ distribution as $n \to \infty$.
2. Suppose that a book of 300 pages contains 200 typographical errors. Use the Poisson approximation to calculate the probability that a given page contains more than one error.
3. Let $\{X_n : \ n \geq 1\}$ be normal r.v. with $\mathrm{E}(X_n) = 1/n$ and $\mathrm{Var}(X_n) = 1$. How are the densities of X_1, X_2 and X_∞? Prove that $X_n \xrightarrow{d} N(0, 1)$ as $n \to \infty$.
4. Let $\{X_n : \ n \geq 1\}$ be r.v. uniform on $(0,\, 1/n)$ and let X be a r.v. such that $\mathrm{P}(X = 0) = 1$. Prove that $X_n \xrightarrow{d} X$ as $n \to \infty$.
5. Let X be uniformly distributed on $(0, 1)$ and for $n = 1, 2, \ldots,$ let X_n be uniformly distributed on $(0, 1 + 1/n)$. Prove that $X_n \xrightarrow{d} X$ as $n \to \infty$.
6. Let $\{X_n : \ n \geq 1\}$ be r.v. uniform on $(1/n,\, 2)$ and let X be uniform on $(0, 2)$. Prove that $X_n \xrightarrow{d} X$ as $n \to \infty$.
7. Let $X_1, X_2, \ldots X_n$ be independent Cauchy random variables. Prove that if $M_n = max\{X_1, X_2, \ldots X_n\}$, then $\pi M_n/n$ converges in distribution to a r.v. with distribution function $H(x) = e^{-1/x}$ for $x \geq 0$.

   Hints:

   - The Cauchy distribution has density $f_X(x) = 1/\big(\pi(1 + x^2)\big)$
   - $e^x = \lim_{n\to\infty}(1 + x/n)^n$
   - $\frac{\pi}{2} = \mathrm{arctg}(x) + \mathrm{arctg}(1/x)$
   - $\mathrm{arctg}(y) = y + o(y)$ as $y \to 0$
8. Let $X_1, \ldots, X_n$ be random samples of a r.v. X which is $\mathsf{Poiss}(\lambda)$. Prove that

$$\frac{\sqrt{n}(\overline{X}_n - \lambda)}{\sqrt{\lambda}} \xrightarrow{d} N(0,\, 1).$$

9. Let X_n be a r.v. such that $\mathrm{P}(X_n = 0) = 1/n = 1 - \mathrm{P}(X_n = 1)$ for all n, and let X be such that $\mathrm{P}(X = 1) = 1$. Prove that $X_n \to X$ in distribution and in probability.
10. Let $S_n = \min(X_1, X_2, \ldots, X_n)$, where $\{X_k : \ k \geq 1\}$ is a sequence of independent r.v., each one uniform on $(0, 1)$. Prove that the sequence $\{S_n : \ n \geq 1\}$ converges to 0 in mean square and in probability.

    Hint: $\int_0^1 t^m (1-t)^n \, dt = (m!\, n!)/(m + n + 1)!$
11. Let $X_1, X_2, \ldots, X_n, \ldots$ be independent r.v. uniform on $(0, 1)$ and let $Z_n = \max(X_1, X_2, \ldots, X_n)$.
    (a) Prove that $\mathrm{P}(Z_n \leq z) = z^n,\ 0 < z < 1$.
    (b) Let $U_n = n(1 - Z_n)$. Prove that the distribution function of U_n converges to the Exp(1) distribution as $n \to \infty$.
    Hint: $\lim_{n\to\infty} (1 + a/n)^n = e^a$.

12. Let $X_1, \ldots, X_n$ be random samples of a r.v. X uniformly distributed on $(0, \theta)$. Let $T_n = 2\,\overline{X}_n$, $V_n = \sqrt{n}\,(T_n - \theta)$, and $W_n = \max(X_1, \ldots, X_n)$. Prove that

    (a) $T_n \xrightarrow{P} \theta$. (Thus, T_n is a consistent estimator of θ.)

    (b) $V_n \xrightarrow{d} N(0,\, \sqrt{\theta^2/3})$.

    (c) $W_n \xrightarrow{P} \theta$.

13. Let $X_1, \ldots, X_n$ be random samples of a r.v. $X \sim \mathsf{Poiss}(\lambda)$. Let $\theta = P(X = 0) = e^{-\lambda}$. Prove that

    (a) $e^{-\overline{X}_n} \xrightarrow{P} \theta$.

    (b) $\sqrt{n}\left(e^{-\overline{X}_n} - \theta\right) \xrightarrow{d} N(0,\, e^{-\lambda}\sqrt{\lambda})$.

14. In a random experiment an event A has probability p. Let A_n be the r.v. that gives the relative frequency of A in n independent repetitions of the experiment. Prove that $\{A_n :\ n \geq 1\}$ converges to p in mean square and almost surely.