Probability and Random Processes

Exercises and Problems

Convergence of sequences of r.v.

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5. Let X be uniformly distributed on $(0,1)$ and for $n = 1, 2, \dots$, let X_n be uniformly distributed on $(0, 1 + 1/n)$. Prove that $X_n \xrightarrow{d} X$ as $n \to \infty$.

   Solution:
   The probability distribution functions of X_n and X are, respectively,

$$F_{X_n}(x) = \begin{cases} 0, & x < 0 \\ \dfrac{n}{n+1}x, & 0 \le x < 1 + \dfrac{1}{n} \\ 1, & x \ge 1 + \dfrac{1}{n} \end{cases} \qquad F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

   Since $F_X(x)$ has no discontinuities we have to prove that $F_{X_n}(x) \to F_X(x)$ for all $x \in \mathbb{R}$.
   If $x < 0$ or $x \ge 2$ the result $F_{X_n}(x) \to F_X(x)$ holds trivially.
   If $0 \le x \le 1$ we have

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \frac{n}{n+1}x = x = F_X(x).$$

   Finally, consider a fixed $x \in (1, 2)$. For all n large enough we have that $x > 1 + 1/n$ and, therefore, $F_{X_n}(x) = 1 = F_X(x)$, completing the proof.

6. Let $\{X_n : n \ge 1\}$ be r.v. uniform on $(1/n, 2)$ and let X be uniform on $(0, 2)$. Prove that $X_n \xrightarrow{d} X$ as $n \to \infty$.

   Solution:
   The probability distribution functions of X_n and X are, respectively,

$$F_{X_n}(x) = \begin{cases} 0, & x < 1/n \\ \dfrac{n}{2n-1}\left(x - \dfrac{1}{n}\right), & \dfrac{1}{n} \le x < 2 \\ 1, & x \ge 2 \end{cases} \qquad F_X(x) = \begin{cases} 0, & x < 0 \\ \dfrac{1}{2}x, & 0 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

   Since $F_X(x)$ has no discontinuities we have to prove that $F_{X_n}(x) \to F_X(x)$ for all $x \in \mathbb{R}$.
   If $x \le 0$ or $x \ge 2$ the result $F_{X_n}(x) \to F_X(x)$ holds trivially. Let $x \in (0, 2)$ be fixed. If n is large enough we have that $x > 1/n$. Hence, for such x,

$$F_{X_n}(x) = \frac{n}{2n-1}\left(x - \frac{1}{n}\right).$$

Therefore,

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \frac{n}{2n-1}\left(x - \frac{1}{n}\right) = \frac{1}{2}x = F_X(x).$$

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10. Let $S_n = \min(X_1, X_2, \ldots, X_n)$, where $\{X_k : \ k \geq 1\}$ is a sequence of independent r.v., each one uniform on $(0, 1)$. Prove that the sequence $\{S_n : \ n \geq 1\}$ converges to 0 in mean square and in probability.
Hint: $\int_0^1 t^m (1-t)^n \, dt = (m! \, n!)/(m+n+1)!$

Solution:
Let $0 < x < 1$. We have

$$F_n(x) = \mathrm{P}(S_n \leq x) = \mathrm{P}(\min(X_1, X_2, \ldots, X_n) \leq x) = 1 - \mathrm{P}(\min(X_1, X_2, \ldots, X_n) > x)$$
$$= 1 - \mathrm{P}(X_1 > x, X_2 > x, \ldots, X_n > x) = 1 - (\mathrm{P}(X_1 > x))^n = 1 - (1-x)^n.$$

Moreover,

$$f_n(x) = F_n'(x) = n(1-x)^{n-1}, \quad 0 < x < 1.$$

To prove that the sequence $\{S_n : \ n \geq 1\}$ converges to 0 in mean square we have to check that $\mathrm{E}(S_n^2) \to 0$ as $n \to \infty$. Indeed,

$$\mathrm{E}(S_n^2) = \int_0^1 x^2 \, n(1-x)^{n-1} \, dx = n \int_0^1 (1-t)^2 \, t^{n-1} \, dt = n\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) \to 0.$$

Convergence in mean square implies convergence in probability. However, let us prove directly that $\{S_n : \ n \geq 1\}$ converges to 0 in probability. We have to prove that, given $\epsilon > 0$, one has $\mathrm{P}(|S_n| > \epsilon) \to 0$ as $n \to \infty$. This is true because

$$\mathrm{P}(|S_n| > \epsilon) = \mathrm{P}(S_n > \epsilon) = (1-\epsilon)^n \to 0.$$

11. Let $X_1, X_2, \ldots, X_n, \ldots$ be independent r.v. uniform on $(0, 1)$ and let $Z_n = \max(X_1, X_2, \ldots, X_n)$.
   (a) Prove that $\mathrm{P}(Z_n \leq z) = z^n, \ 0 < z < 1$.
   (b) Let $U_n = n(1 - Z_n)$. Prove that the distribution function of U_n converges to the Exp(1) distribution as $n \to \infty$.
   Hint: $\lim_{n\to\infty} (1 + a/n)^n = e^a$.

   Solution:
   (a) If $0 < z < 1$ we have

   $$\mathrm{P}(Z_n \leq z) = \mathrm{P}(\max(X_1, X_2, \ldots, X_n) \leq z) = \mathrm{P}(X_1 \leq z, X_2 \leq z, \ldots, X_n \leq z) = (\mathrm{P}(X_1 \leq z))^n = z^n.$$

   (b) We have to prove that

   $$F_{U_n}(x) = \mathrm{P}(U_n \leq x) \to \begin{cases} 1 - e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

   Since U_n takes positive values, $\mathrm{P}(U_n \leq x) = 0$ if $x < 0$. Let $x > 0$ be fixed. We have

   $$\mathrm{P}(U_n \leq x) = \mathrm{P}(n(1 - Z_n) \leq x) = \mathrm{P}\left(1 - Z_n \leq \frac{x}{n}\right) = \mathrm{P}\left(Z_n \geq 1 - \frac{x}{n}\right) = 1 - \mathrm{P}\left(Z_n \leq 1 - \frac{x}{n}\right).$$

   Let $z = 1 - x/n$. If $n > x$ then $0 < z < 1$ and, hence,

   $$\mathrm{P}(U_n \leq x) = 1 - \left(1 - \frac{x}{n}\right)^n \to 1 - e^{-x}.$$

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