Probability and Random Processes

Exercises and Problems

Generating and characteristic functions. Gaussian random vectors

1. Let X and Y be independent discrete r.v. such that

$$P(X = 0) = \frac{1}{2}, \quad P(X = 1) = \frac{1}{4}, \quad P(X = 2) = \frac{1}{8}, \quad P(X = 3) = \frac{1}{8},$$

$$P(Y = 1) = \frac{1}{3}, \quad P(Y = 3) = \frac{2}{3}.$$

Use generating functions to find the probability function of $X + Y$.

2. Let X and Y be independent discrete r.v. such that

$$P(X = 0) = \frac{1}{2}, \quad P(X = 1) = \frac{1}{4}, \quad P(X = 2) = \frac{1}{4}$$

$$P(Y = 1) = \frac{1}{4}, \quad P(Y = 3) = \frac{3}{4}.$$

Find the moment generating function of $X - Y$.

3. Two balls are picked at random from an urn that contains five balls, two of which are white and the rest are black. Let X be the number of white balls in the selection.
   (a) Find the characteristic function of X and calculate $E(X)$ and $Var(X)$.
   (b) Two additional balls are picked from another identical urn. Find the characteristic function of the total number Y of white balls in the two extractions.

4. (a) Calculate the characteristic function of an exponential r.v. with parameter λ.
   (b) Prove that

$$M_X(\omega) = \frac{e^{3i\omega - 2\omega^2}}{1 + i\omega}$$

   is a characteristic function and find $E(X)$ and $Var(X)$.

5. (a) Let X, Y be random variables with characteristic functions M_X and M_Y respectively, and joint characteristic function M_{XY}. A necessary and sufficient condition for X and Y to be independent is $M_{XY}(\omega_1, \omega_2) = M_X(\omega_1)M_Y(\omega_2)$. Prove the necessity of the condition.
   (b) Let X and Y be independent gaussian $N(0,1)$ random variables. Use characteristic functions to prove that $S = X + Y$ and $R = X - Y$ are independent. (Check that the above sufficient condition holds for S and R.)

6. Let X_1, X_2 and X_3 be independent r.v. uniform on $[-1, 1]$. Find the characteristic function of $X = \sum_{i=1}^{N} X_i$ where N is a r.v. independent of X_1, X_2, X_3, such that $P(N = 1) = P(N = 2) = P(N = 3) = 1/3$.

7. A r.v. X is called symmetric if X and $-X$ have the same probability distribution. Prove that X is symmetric if and only if the imaginary part of its characteristic function is 0.

8. Let X be a non-negative, integer-valued random variable satisfying:

$$P(X = 0) = \frac{2}{3} P(X = 1),$$

$$P(X = 2n) = \frac{1}{2} P(X = 2n - 1) = \frac{2}{3} P(X = 2n + 1), \quad n \geq 1.$$

Compute its probability generating function.

9. Use the moment generating function to compute $\mathrm{E}(X^4)$, where $X \sim \mathsf{Bin}(n, p)$.

10. The random variable X has the property that $\mathrm{E}(X^n) = 3^n/(n+1)$, $n = 1, 2, \ldots$ Find the unique distribution of X having these moments.

11. Use moment generating functions to prove that, if a random variable X has density function

$$f_X(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty,$$

then X can be written as $X = Y - Z$, where Y and Z are independent, exponentially distributed random variables.

12. The number of cars passing a road crossing during a day follows a Poisson distribution with pararmeter λ. The number of persons in each car is a $\mathsf{Poiss}(\alpha)$ random variable. Find the probability generating function of the total number of persons, N, passing the road crossing during a day. Find the mean and variance of N.

13. Let $X \sim \mathsf{N}(0, 1)$ be a standard normal r.v.

    (a) Find the density of $Y = X^2$.

    (b) The characteristic function of X^2 is $(1 - 2i\omega)^{-1/2}$. Find the characteristic function of $Z = X_1^2 + X_2^2 + \cdots + X_n^2$ where $X_1, X_2, \ldots, X_n$ are all standard normal and independent.

14. Let $Z = S + N$ where $S \sim \mathsf{N}(m_S, \sigma_S^2)$ and $N \sim \mathsf{N}(0, \sigma_N^2)$ are independent. Find the joint characteristic function of S and Z.

15. Let X_1, X_2 be gaussian r.v. with mean 0 and covariance matrix

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

If $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$, find the characteristic function of (Y_1, Y_2) and the marginal density of Y_1.

16. A 2-dimensional symmetric random walk is a sequence of points $\{(X_n, Y_n) : \ n \geq 0\}$ defined in the following way. If $(X_n, Y_n) = (x, y)$, then (X_{n+1}, Y_{n+1}) is, with equal probability, one of the four points $(x+1, y)$, $(x-1, y)$, $(x, y+1)$, $(x, y-1)$. We assume that $(X_0, Y_0) = (0, 0)$.

    (a) Prove that $E(X_n^2 + Y_n^2) = n$.

    (b) Find the characteristic function of (X_n, Y_n).

17. Let X and Y be continuous r.v. and let $Z = aX + bY$ where $a, b \in \mathbb{R}$.

    (a) Prove that if the density f_Z is known for all $a, b \in \mathbb{R}$, then the joint density f_{XY} is univocally determined.

    (b) Prove that if the r.v. Z is gaussian for all $a, b \in \mathbb{R}$, then X and Y are jointly gaussian.

18. Let Y_1 and Y_2 be jointly gaussian r.v. such that $\mathrm{E}(Y_1) = 1$, $\mathrm{E}(Y_2) = -1$, $\mathrm{Var}(Y_1) = 4$, $\mathrm{Var}(Y_2) = 1$ and $\rho = 1/2$. Let N be also gaussian with $\mathrm{E}(N) = 0$, $\mathrm{Var}(N) = 2$ and independent of (Y_1, Y_2). If $X = Y_1 - Y_2 + N$, prove that (X, Y_1) is a gaussian vector and compute its parameters.

19. (a) Let X, Y be random variables with characteristic functions M_X and M_Y respectively, and joint characteristic function M_{XY}. A necessary and sufficient condition for X and Y to be independent is $M_{XY}(\omega_1, \omega_2) = M_X(\omega_1)M_Y(\omega_2)$. Prove the necessity of the condition.

    (b) Let X and Y be independent gaussian $\mathsf{N}(0,1)$ random variables. Use characteristic functions to prove that $S = X + Y$ and $R = X - Y$ are independent. (Check that the above sufficient condition holds for S and R.)

20. The independent r.v. X_i $(i \geq 1)$ are all gaussian with mean 1 and variance 2. Find

$$\mathrm{P}(X_{n+1} > X_1 + X_2 + \cdots + X_n).$$

Express the result in terms of $F_{\mathsf{N}(0,1)}$.

21. Let X, Y, Z be jointly gaussian r.v. such that $m_X = m_Y = 0$, $m_Z = -1$, $\sigma_X^2 = \sigma_Y^2 = \sigma_Z^2 = 1$. Moreover, we know that X and Y are uncorrelated but the correlation coefficient of X and Z is $1/2$ and the correlation coefficient of Y and Z is $3/4$. Calculate the variance of $X + 2Y - 3Z$.

22. Prove that the standard normal distribution $\mathsf{N}(0,1)$ has all its moments of odd order equal to 0 while the moments of even order are given by

$$\mu_{2j} = \frac{(2j)!}{2^j(j!)} \quad j = 0, 1, 2, \ldots$$

23. Let X and Y be independent gaussian $\mathsf{N(0,1)}$ random variables. Prove that $Z = X + Y$ and $W = X - Y$ are independent.

24. Let X be a gaussian r.v. with expectation vector $m = (1, -1, 2)^t$ and covariance matrix

$$K = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Which of the following r.v. are independent?: X_1 and X_2; X_1 and X_3; X_2 and X_3; X_1 and $X_1 + 3X_2 - 2X_3$.