Probability and Random Processes

Exercises and Problems

Branching Processes. Distributions with random parameters.

1. (A. Gut, III.25) Consider a branching process with one ancestor. Suppose that the generating function of the offspring distribution is
$$G(s) = \frac{p^2}{(1-qs)^2},$$
where $0 < p = 1 - q < 1$. What is the probability

   (a) of extinction?

   (b) that the process is extinct in generation number 2 (i.e., the ancestor does not have any grandchildren)?

   Answer:
   (a) If $0 < p < 2/3$, then $d = \frac{1}{q} - \frac{1}{2} - \sqrt{\frac{1}{q} - \frac{3}{4}}$. If $p \geq 2/3$, then $d = 1$.
   (b) $d_2 = p^2/(1-p^2q)^2$.

2. (A. Gut, III.26) The following model can be used to describe the number of women (mothers and daughters) in a given area. The number of mothers is a random variable $X \sim \mathsf{Po}(\lambda)$. Independently of the others, every mother gives birth to a $\mathsf{Po}(\mu)$-distributed number of daughters. Let Y be the total number of daughters and hence $Z = X + Y$ be the total number of women in the area.

   (a) Find the generating function of Z.

   (b) Compute $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$.

   Answer:
   (a) $G_Z(s) = \exp\left(\lambda\left(se^{\mu(s-1)} - 1\right)\right)$.
   (b) $\mathrm{E}(Z) = \lambda(1+\mu)$; $\mathrm{Var}(Z) = \lambda(1 + 3\mu + \mu^2)$.

3. (A. Gut, III.34) Let X be the number of coin tosses until heads is obtained. Suppose that the probability of heads is unknown in the sense that we consider it to be a random variable $Y \sim \mathsf{U}(0, 1)$. Find the distribution of X.

   Answer: $\mathrm{P}(X = k) = \frac{1}{k(k+1)}$.