

Probability and Random Processes

Exercises and Problems

Generating and characteristic functions. Gaussian random vectors

1. $G_{X+Y}(s) = \frac{1}{6}s + \frac{1}{12}s^2 + \frac{3}{8}s^3 + \frac{5}{24}s^4 + \frac{1}{12}s^5 + \frac{1}{12}s^6$

Hence, if $Z = X + Y$ then $P(Z = 0) = 0$, $P(Z = 1) = \frac{1}{6}$, $P(Z = 2) = \frac{1}{12}$, $P(Z = 3) = \frac{3}{8}$, $P(Z = 4) = \frac{5}{24}$, $P(Z = 5) = \frac{1}{12}$, $P(Z = 6) = \frac{1}{12}$

2. $\phi_{X-Y}(t) = \frac{1}{16} + \frac{3e^{-3t}}{8} + \frac{3e^{-2t}}{16} + \frac{5e^{-t}}{16} + \frac{e^t}{16}$

3. $M_X(\omega) = \frac{1}{10}(3 + 6e^{i\omega} + e^{i2\omega})$; $M_Y(\omega) = \frac{1}{100}(3 + 6e^{i\omega} + e^{i2\omega})^2$

4. Answer: $E(X) = 2$; $\text{Var}(X) = 5$

5.

6. $M_X(\omega) = \frac{1}{3} \left(\frac{\sin \omega}{\omega} + \frac{\sin^2 \omega}{\omega^2} + \frac{\sin^3 \omega}{\omega^3} \right)$

7.

8. $G_X(s) = \frac{2 + 3s}{5(4 - 3s^2)}$, $|s| < \frac{2}{\sqrt{3}}$

9. $np + 14 \binom{n}{2} p^2 + 36 \binom{n}{3} p^3 + 24 \binom{n}{4} p^4$

10. $X \sim \text{Unif}[0, 3]$

11.

12. If X is the number of cars and Y the number of persons in each car, then $G_N(s) = G_X(G_Y(s)) = \exp(\lambda(e^{\alpha(s-1)} - 1))$. Thus, $E(N) = \lambda\alpha$, $\text{Var}(N) = \lambda\alpha(\alpha + 1)$.

13. $f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}y} e^{-\frac{1}{2}y}, & y > 0 \\ 0, & y < 0 \end{cases}$; $M_Z(\omega) = (1 - 2i\omega)^{-n/2}$

14. $M_{SZ}(\omega_1, \omega_2) = e^{i(\omega_1 + \omega_2)m_S - \frac{1}{2}\sigma_S^2(\omega_1 + \omega_2)^2} e^{-\frac{1}{2}\omega_1^2\sigma_N^2}$

15. $M_Y(\omega_1, \omega_2) = e^{-\frac{1}{2}3\omega_1^2} e^{-\frac{1}{2}\omega_2^2} = M_{Y_1}(\omega_1)M_{Y_2}(\omega_2)$, Y_1 and Y_2 are independent.

$$f_{Y_1}(y) = \frac{1}{\sqrt{2\pi}\sqrt{3}} e^{-\frac{1}{2}\frac{y^2}{3}}$$

16. $M_{X_n Y_n}(\omega_1, \omega_2) = \frac{1}{2^n} (\cos \omega_1 + \cos \omega_2)^n$

17.

18. $E(X) = 2$, $\text{Var}(X) = 5$, $\mu_{11;XY_1} = 3$

19.

20. $1 - F_{N(0,1)}\left(\frac{n-1}{\sqrt{2(n+1)}}\right)$

21. 2

22.

23.

24.