

Applications of on-line prediction

in telecommunication problems

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based on joint work with

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Outline

- On-line prediction;
- Some algorithmic issues;
- An application to lossy data compression without delay;
- An application to a routing problem.

Randomized prediction

A game between forecaster and environment.

At each round t , the forecaster chooses an action $I_t \in \{1, \dots, N\}$;

the environment chooses an action $y_t \in \mathcal{Y}$;

the forecaster suffers loss $\ell(I_t, y_t) \in [0, 1]$.

The goal is to minimize the *cumulative excess loss*

$$\frac{1}{n} \left(\sum_{t=1}^n \ell(I_t, y_t) - \min_{i \leq N} \sum_{t=1}^n \ell(i, y_t) \right) .$$

The forecaster may randomize. At time t chooses a probability distribution $\mathbf{p}_t = (p_{1,t}, \dots, p_{N,t})$

and plays action i with probability $p_{i,t}$.

Actions are often called “experts”.

Randomized prediction

This and related models have been studied in

- game theory: playing repeated games;
- information theory: gambling and data compression;
- statistics: sequential decisions;
- statistical learning theory: on-line learning;

The simplest model assumes that after each round, the losses $\ell(i, y_t)$ ($i = 1, \dots, N$) are revealed (*full information*).

In this model **Hannan** (1957) showed that the forecaster has a strategy such that

$$\frac{1}{n} \left(\sum_{t=1}^n \ell(I_t, y_t) - \min_{i \leq N} \sum_{t=1}^n \ell(i, y_t) \right) \rightarrow 0 \quad a.s.$$

Hannan consistency: basic ideas

Obviously, the forecaster must randomize.

$$\ell(\mathbf{p}_t, y_t) = \sum_{i=1}^N p_{i,t} \ell(i, y_t) = \mathbf{E}_t \ell(I_t, y_t)$$

denotes the “expected” loss of the forecaster.

By martingale convergence,

$$\frac{1}{n} \left(\sum_{t=1}^n \ell(I_t, y_t) - \sum_{t=1}^n \ell(\mathbf{p}_t, y_t) \right) = O_P(n^{-1/2})$$

so it suffices to study

$$\frac{1}{n} \left(\sum_{t=1}^n \ell(\mathbf{p}_t, y_t) - \min_{i \leq N} \sum_{t=1}^n \ell(i, y_t) \right)$$

Weighted average prediction

Idea: assign a higher probability to better-performing actions.

A popular choice is

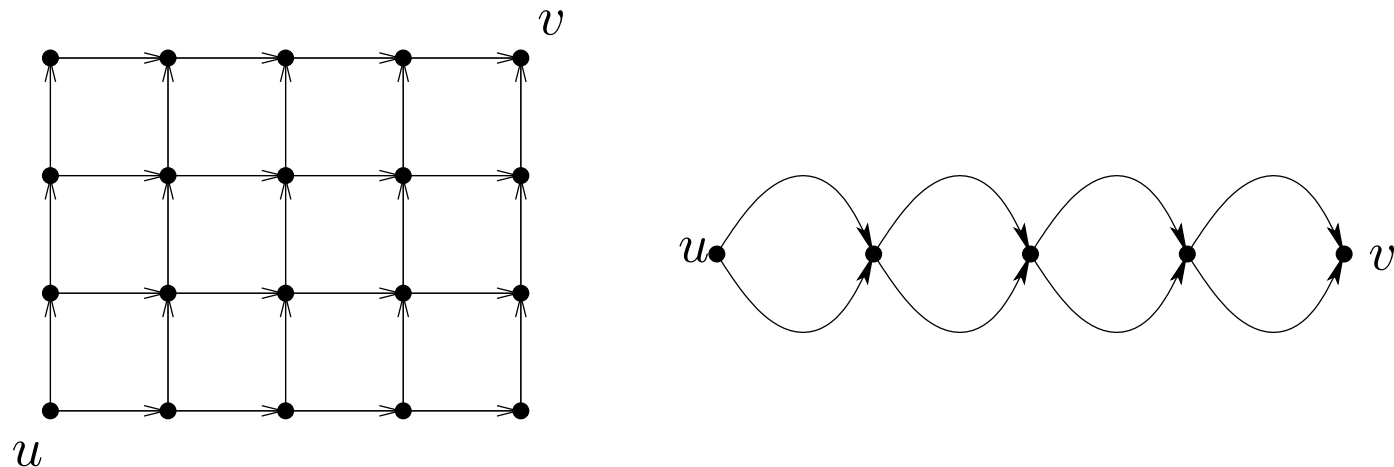
$$p_{i,t} = \frac{\exp\left(-\eta \sum_{s=1}^t \ell(i, y_s)\right)}{\sum_{k=1}^N \exp\left(-\eta \sum_{s=1}^t \ell(k, y_s)\right)} \quad i = 1, \dots, N .$$

where $\eta > 0$. Then, with $\eta = \sqrt{8 \ln N/n}$,

$$\begin{aligned} \frac{1}{n} \left(\sum_{t=1}^n \ell(\mathbf{p}_t, y_t) - \min_{i \leq N} \sum_{t=1}^n \ell(i, y_t) \right) &\leq \frac{\ln N}{n\eta} + \frac{\eta}{8} \\ &= \sqrt{\frac{\ln N}{2n}} \end{aligned}$$

If N is large, the algorithm is not feasible. There is hope for structured classes of experts.

The shortest path problem



Given a DAG (V, E) and vertices u and v , a path from u to v is identified by a binary vector $\mathbf{i} \in \{0, 1\}^{|E|}$.

For each $t = 1, \dots, n$, the forecaster chooses a path \mathbf{I}_t . A loss $\ell_{e,t} \in [0, 1]$ is assigned to each $e \in E$.

The loss of a path \mathbf{i} is the sum of losses over the edges

$$\ell(\mathbf{i}, Y_t) = \mathbf{i} \cdot \mathbf{l}_t .$$

Efficient implementation

The number of paths (experts) is exponentially large. However, the weighted average forecaster can be computed efficiently. For each path \mathbf{i} , we write

$$p_{\mathbf{i},t} = \mathbf{P}_t[\mathbf{I}_t = \mathbf{i}] = \prod_{k=1}^{K_{\mathbf{i}}} \mathbf{P}_t[v_{\mathbf{I}_t,k} = v_{\mathbf{i}_k} \mid v_{\mathbf{I}_t,k-1} = v_{\mathbf{i},k-1}, \dots, v_{\mathbf{I}_t,0} = u].$$

It can be seen that, if $(\hat{w}, w) \in E$,

$$\mathbf{P}_t[v_{\mathbf{I}_t,k} = w \mid v_{\mathbf{I}_t,k-1} = \hat{w}, \dots, v_{\mathbf{I}_t,0} = u] = e^{-\eta L_{(\hat{w}, w), t-1}} \frac{G_{t-1}(w)}{G_{t-1}(\hat{w})}$$

where, for any vertex w ,

$$G_t(w) = \sum_{\mathbf{i} \in \mathcal{P}_w} e^{-\eta \sum_{e \in \mathbf{i}} L_{e,t}}.$$

The algorithm computes the weighted average forecaster over all paths such that it requires $O(|E|)$ operations at each time. The expected regret is bounded by

$$\sum_{t=1}^n \bar{\ell}(\mathbf{p}_t, Y_t) - \min_{\mathbf{i}} \sum_{t=1}^n \ell(\mathbf{i}, Y_t) \leq K \sqrt{\frac{n \ln M}{2}}$$

where M is the number of paths from u to v and K is the length of the longest path.

Lossy source coding problem

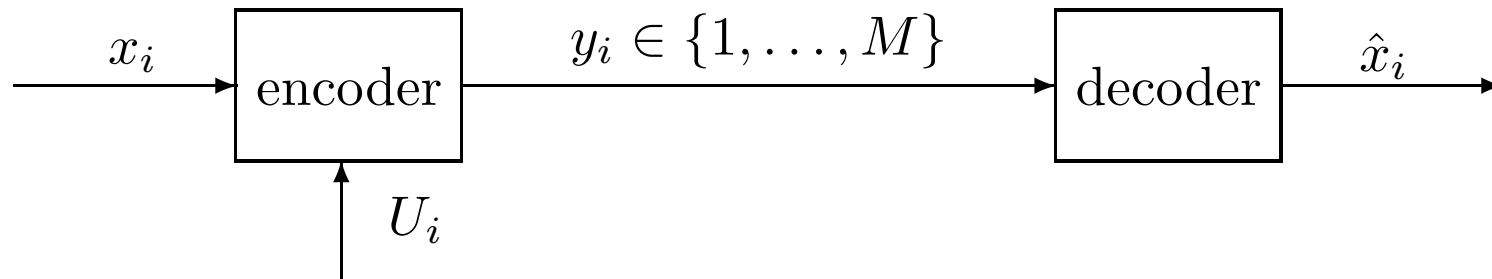
Asymptotically optimal coding of a source sequence with

- zero delay (sequential encoding and decoding);
- no probabilistic assumptions on the source;
- asymptotically the same performance as the best scalar quantizer matched to the entire source sequence (for any sequence).

Goals:

- solve the above problem with low complexity;
- determine the achievable rate of convergence (also for probabilistic sources)

Zero-Delay Sequential Source Coding



- Source sequence: $x_i \in [0, 1], i = 1, 2, \dots$;
- rate $R = \log M$ code:
 - Encoder: $f_i : [0, 1]^i \rightarrow \{1, 2, \dots, M\}$;
 - Decoder: $g_i : \{1, 2, \dots, M\}^i \rightarrow [0, 1]$;

$$y_i = f_i(x^i) = f_i(x^i, U^i)$$

$$\hat{x}_i = g_i(y^i)$$

- Randomizing sequence: U_i , i.i.d. uniform on $[0, 1]$;

Performance

Expected normalized cumulative squared distortion:

$$\mathbf{E} \left[\frac{1}{n} \sum_{i=1}^n (x_i - \hat{x}_i)^2 \right]$$

Goal: Perform (essentially) as well as the best scalar quantizer matched to the entire sequence $x^n = (x_1, \dots, x_n)$.

Expected distortion redundancy with respect to the class of scalar quantizers:

$$\rho_n = \sup_{x^n} \left(\mathbf{E} \left[\frac{1}{n} \sum_{i=1}^n (x_i - \hat{x}_i)^2 \right] - \min_{Q \in \mathcal{Q}} \frac{1}{n} \sum_{i=1}^n (x_i - Q(x_i))^2 \right)$$

where \mathcal{Q} is the set of M -level scalar quantizers.

Choosing Q_k

Let \mathcal{Q}_K ($K \sim n^{1/3}$ for minimum distortion) be the set of M -level nearest neighbor scalar quantizers with code points in the set

$$C^{(K)} = \{1/(2K), 3/(2K), \dots, (2K - 1)/(2K)\}.$$

Q_k is chosen randomly from \mathcal{Q}_K according to the probabilities

$$p_k(Q) = \mathbf{P}\{Q_k = Q\} = \frac{e^{-\eta \sum_{t=1}^{kl} (x_t - Q(x_t))^2}}{\sum_{\hat{Q} \in \mathcal{Q}_K} e^{-\eta \sum_{t=1}^{kl} (x_t - \hat{Q}(x_t))^2}}$$

$$\text{Performance: } \rho_n \leq C_1 \sqrt{\frac{l \log K}{n}} + C_2 \frac{\log K}{l} + \frac{1}{K}$$

where $C_1, C_2 > 0$ are constants depending only on M .

Straightforward implementation: need to compute the cumulative distortion for all $Q \in \mathcal{Q}_K$, i.e., for about $K^M \sim n^{M/3}$ quantizers.

Efficient implementation

The “shortest path” algorithm can be used to construct a zero-delay coding scheme of rate $R = \log M$ such that for any sequence $x^n \in [0, 1]^n$

$$\rho_n \leq C_1 \sqrt{\frac{l \log K}{n}} + C_2 \frac{\log K}{l} + \frac{3}{K}$$

where $C_1, C_2 > 0$ are constants depending only on M , with

- $O(MK^2n/l)$ computational complexity; and
- $O(K^2)$ memory requirement.

Linear computational complexity $O(M)$ per time instant and $O(n^{1/2})$ memory requirement with $O(n^{-1/4} \sqrt{\log n})$ distortion redundancy

Tracking the best expert

(Herbster and Warmuth, 1998). Given any sequence i_1, \dots, i_n of actions from $\{1, \dots, N\}$, define $\text{size}(i_1, \dots, i_n) = \sum_{t=2}^n \mathbb{I}_{i_{t-1} \neq i_t}$. The *tracking regret* is

$$\max \left(\sum_{t=1}^n \bar{\ell}(\mathbf{p}_t, Y_t) - \sum_{t=1}^n \ell(i_t, Y_t) \right)$$

over all sequences with $\text{size}(i_1, \dots, i_n) \leq m$.

The number of “meta” experts is $\sum_{k=0}^m \binom{n-1}{k} N(N-1)^k$. Simple weighted average would give a regret bounded by

$$\sqrt{\frac{n}{2} \left((m+1) \ln N + m \ln \frac{e(n-1)}{m} \right)}$$

Fixed share forecaster

Parameters: Real numbers $\eta > 0$ and $0 \leq \alpha = m/(n - 1)$.

Initialization: $w_0 = (1/N, \dots, 1/N)$. For each round $t = 1, 2, \dots$

(1) draw an action I_t from $\{1, \dots, N\}$ according to the distribution

$$p_{i,t} = \frac{w_{i,t-1}}{\sum_{j=1}^N w_{j,t-1}} \quad i = 1, \dots, N$$

(2) obtain Y_t and compute

$$v_{i,t} = w_{i,t-1} e^{-\eta \ell(i, Y_t)} \quad \text{for each } i = 1, \dots, N$$

(3) let

$$w_{i,t} = \alpha \frac{W_t}{N} + (1 - \alpha)v_{i,t} \quad \text{for each } i = 1, \dots, N$$

where $W_t = v_{1,t} + \dots + v_{N,t}$.

Tracking the shortest path

Goal: predict as well as the best path that can change m times.

$\sum_{k=0}^m \binom{n-1}{k} M(M-1)^k$ “meta” experts. (M is the number of paths.)

The fixed share forecaster requires computation proportional to M .

We need efficient computation of the fixed share algorithm.

Fixed share forecaster revisited

Initialization: For $t = 1$, choose I_1 uniformly from the set $\{1, \dots, N\}$. For each round $t = 2, \dots, n$

(1) Draw τ_t randomly according to the distribution

$$\mathbf{P}_t[\tau_t = t'] = \begin{cases} \frac{(1-\alpha)^{t-1} Z_{1,t-1}}{NW_{t-1}} & \text{for } t' = 1 \\ \frac{\alpha(1-\alpha)^{t-t'} W_{t'-1} Z_{t',t-1}}{NW_{t-1}} & \text{for } t' = 2, \dots, t \end{cases}$$

where we define $Z_{t,t-1} = N$.

(2) Given $\tau_t = t'$, choose I_t randomly according to the probabilities

$$\mathbf{P}_t[I_t = i | \tau_t = t'] = \begin{cases} \frac{e^{-\eta \sum_{s=t'}^{t-1} \ell(i, Y_s)}}{Z_{t',t-1}} & \text{for } t' = 1, \dots, t-1 \\ \frac{1}{N} & \text{for } t' = t \end{cases}$$

Tracking the shortest path

The alternative fixed share algorithm can be implemented such that, at time t , computing the prediction I_t requires time $O(tK|E| + t^2)$. The expected tracking regret of the algorithm satisfies

$$K \sqrt{\frac{n}{2} \left((m+1) \ln M + m \ln \frac{en}{m} \right)}$$

for all sequences of paths $\mathbf{i}_1, \dots, \mathbf{i}_n$ such that $\text{size}(\mathbf{i}_1, \dots, \mathbf{i}_n) \leq m$ where M is the number of all paths in the graph between vertices u and v and K is the length of the longest path.

Application to zero-delay coding

An efficiently computable ($O(M)$ computations per time instant) zero-delay coding scheme can be constructed whose average distortion is at most as large as the best code that can change scalar quantizers at most m times plus a redundancy of the order of $\sqrt{mn}^{-1/6}$.

Multi-armed bandit problem

(Auer, Cesa-Bianchi, Freund, and Schapire, 2002.) The forecaster only observes his/her own loss.

Losses of experts have to be “estimated”.

Let $g(i, Y_t) = 1 - \ell(i, Y_t)$. Gains are estimated by

$$\tilde{g}(i, Y_t) = \begin{cases} g(i, Y_t)/p_{i,t} & \text{if } I_t = i \\ 0 & \text{otherwise} \end{cases} \quad i = 1, \dots, N .$$

Note that $\mathbf{E}[\tilde{g}(i, Y_t) \mid I_1, \dots, I_{t-1}] = g(i, Y_t)$, and therefore $\tilde{g}(i, Y_t)$ is an unbiased estimate of $g(i, Y_t)$.

A strategy for the multi-armed bandit problem

Initialization: $w_{i,0} = 1$ and $p_{i,1} = 1/N$ for $i = 1, \dots, N$.

- (1) Select an action $I_t \in \{1, \dots, N\}$ according to the probability distribution \mathbf{p}_t .
- (2) Calculate the estimated gains

$$g'(i, Y_t) = \tilde{g}(i, Y_t) + \frac{\beta}{p_{i,t}} = \begin{cases} (g(i, Y_t) + \beta)/p_{i,t} & \text{if } I_t = i \\ \beta/p_{i,t} & \text{otherwise} \end{cases}$$

- (3) Update the weights $w_{i,t} = w_{i,t-1} e^{\eta g'(i, Y_t)}$
- (4) Calculate the updated probability distribution

$$p_{i,t+1} = (1 - \gamma) \frac{w_{i,t}}{\sum_{j=1}^N w_{j,t}} + \frac{\gamma}{N}, \quad i = 1, \dots, N.$$

Performance bound

The regret of the algorithm satisfies, with probability $1 - \delta$,

$$\widehat{L}_n - \min_{i=1,\dots,N} L_{i,n} \leq \frac{11}{2} \sqrt{nN \ln(N/\delta)} + \frac{\ln N}{2} .$$

Bound depends badly on the number of actions. Cannot be improved, in general.

Bandit version of shortest path: a routing problem

A packet is sent through a network (graph), and at each edge the transmission suffers a delay that can vary arbitrarily. We want to choose the route such that the total delay is not much larger than that of the best constant route.

We only receive information about the delays of the packets sent, but don't know anything about the other edges.

The general bandit algorithm needs to be modified to exploit the special structure of the class of experts of all paths.

The probability of certain paths needs to be kept large.

We obtain an algorithm, computable in time quadratic in $|E|$ whose regret is, with probability $1 - \delta$,

$$\sum_{t=1}^n \ell(\mathbf{I}_t, Y_t) - \min_{\mathbf{i}} \sum_{t=1}^n \ell(\mathbf{i}, Y_t) \leq \sqrt{nK^3|E| \ln(M/\delta)} .$$