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Pseudo-normal form near saddle-center or saddle-focus equilibria

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Abstract

In this paper we introduce the pseudo-normal form, which generalizes the notion of normal form around an equilibrium. Its convergence is proved for a general analytic system in a neighborhood of a saddle-center or a saddle-focus equilibrium point. If the system is Hamiltonian or reversible, this pseudo-normal form coincides with the Birkhoff normal form, so we present a new proof in these celebrated cases. From the convergence of the pseudo-normal form for a general analytic system several dynamical consequences are derived, like the existence of local invariant objects.

MSC: 34C20; 34C14

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1. Introduction and main results

Since normal forms were introduced by Poincaré they have become a very useful tool to study the local qualitative behavior of dynamical systems around equilibria, see for instance [1,3,8] and references therein. In a few words, given a system

$$\dot{X} = F(X) = \Lambda X + \widehat{F}(X), \qquad (1)$$

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around an equilibrium $X = 0$, where $\widehat{F}(X)$ denotes terms of order at least 2 in X, a general normal form procedure consists on looking for a (formal power series close to the identity) transformation $X = \Phi(\chi) = \chi + \widehat{\Phi}(\chi)$ in such a way that the new system $\dot{\chi} = \Phi^* F(\chi) =: N(\chi) = \Lambda\chi + \widehat{N}(\chi)$ becomes in *normal form*, that is, when $\widehat{N}$ contains only the so-named *resonant terms*, monomials whose powers are intimately related to the vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ formed by the eigenvalues of the matrix Λ of system (1).

In this work, we will focus our attention on analytic vector fields and will be specially concerned with the convergence of the *normalizing transformation* Φ. There are two well-known cases where the convergence of the normalizing transformation follows just from the properties of the vector of characteristic exponents λ (see, for instance, [1, Chapter 5, Section 24]):

(i) when λ belongs to the Poincaré domain, that is, the convex hull of the set $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ in the complex plane does not contain the origin;
(ii) when λ belongs to the complementary of this domain, the so-called Siegel's domain, and satisfies a Diophantine condition.

In the first case, the Theorem of Poincaré–Dulac ensures the convergence of a normalizing transformation conjugating the original system to a system having only a finite number of resonant terms. In the second case, the Diophantine condition permits to bound the small divisors appearing in the normalizing transformation and its convergence is also derived (Siegel's Theorem). The original system is conjugated to its linear part.

Notice that in both cases of convergence the normal form is a *polynomial* or, in other words, the number of resonant terms is finite. However, non polynomial normal forms do arise in some important families of dynamical systems, like the Hamiltonian or the reversible ones, where the characteristic exponents always belong to the Siegel's domain since they come in pairs $\{\pm\lambda\}$. In these cases, convergence results depend not only on the location of the characteristic exponents and their arithmetical properties but also on the kind of formal normal form they exhibit.

In 1971, Bruno (see [2, Chapter II, Sections 3, 4]) provided sufficient and, in some particular sense, necessary conditions ensuring this convergence. He denominated them *condition* ω and *condition* A. The condition ω depends on arithmetic properties of the vector of characteristic exponents λ, and can be checked explicitly. On the contrary, condition A imposes a strong restriction on the normal form forcing it (up to all order!) to depend only on one or two scalar functions.

We refer the reader to Bruno's paper [2, pp. 173–175] for a detailed account of these conditions. For the purpose of this paper, it is enough to notice that there are very few cases where the fulfillment of condition A follows from the nature of the original system. The most famous case is provided by the framework of the Hamiltonian systems, where the normal form is called the *Birkhoff normal form* (BNF in short). Among them, condition ω is trivially satisfied when there are no small divisors between the main characteristic exponents, but this only happens for Hamiltonian systems with one or two degrees of freedom.

Indeed, consider a 2-degrees of freedom Hamiltonian system and denote by $\{\pm\lambda_1, \pm\lambda_2\}$ its characteristic exponents at the origin. The condition for the non existence of small divisors between λ_1 and λ_2 is that $\lambda_1/\lambda_2 \notin \mathbb{R}$, and this condition is satisfied only when the origin is

- a *saddle-focus*, if $\{\pm\lambda_1, \pm\lambda_2\} = \{\pm\lambda \pm \mathrm{i}\alpha\}$ with $\lambda > 0$, $\alpha > 0$, or
- a *saddle-center*, if $\{\pm\lambda_1, \pm\lambda_2\} = \{\pm\lambda, \pm\mathrm{i}\alpha\}$ and $\lambda > 0$, $\alpha > 0$.

In these two cases, N can be written as

$$\text{(a)} \quad N = \begin{pmatrix} \xi a_1(\xi\eta, \mu\nu) \\ -\eta a_1(\xi\eta, \mu\nu) \\ \mu a_2(\xi\eta, \mu\nu) \\ -\nu a_2(\xi\eta, \mu\nu) \end{pmatrix}, \quad \text{(b)} \quad N = \begin{pmatrix} \xi a_1(\xi\eta, \mu^2+\nu^2) \\ -\eta a_1(\xi\eta, \mu^2+\nu^2) \\ \nu a_2(\xi\eta, \mu^2+\nu^2) \\ -\mu a_2(\xi\eta, \mu^2+\nu^2) \end{pmatrix}, \tag{2}$$

respectively, where $a_j(0,0) = \lambda_j$, $j = 1,2$ and $\chi = (\xi, \eta, \mu, \nu) \in \mathbb{C}^4$.

The existence of an analytic transformation leading an analytic Hamiltonian system in the neighborhood of a saddle-focus or a saddle-center into BNF was provided in 1958 by Moser [14], extending the famous Lyapunov theorem [11]. (Recently, a new proof of this theorem has been provided by Giorgilli [9] putting special emphasis on the Hamiltonian character of the system—a characteristic which does not appear in Moser's proof.)

At this point, it seems natural to wonder about the convergence of a normalizing transformation Φ in the case of a general system. The analogy with the Hamiltonian case suggests to consider two-dimensional and four-dimensional systems with characteristic exponents at the equilibrium point of the form (i) $\pm\lambda$, $\lambda \in \mathbb{C}\backslash\{0\}$ and (ii) $\{\pm\lambda_1, \pm\lambda_2\}$, respectively, to avoid small divisors. Case (i) was studied in [5]. The aim of the present work is to deal with case (ii), a general analytic system (1) with a saddle-focus or a saddle-center equilibrium point at the origin.

Let us be more precise. As it has been said, it is well-known in the saddle-focus or saddle-center Hamiltonian cases the existence of a convergent transformation $X = \Phi(\chi)$ leading system (1) into BNF, that is, the transformed system being of the form

$$\dot{\chi} = (\Phi^* F)(\chi) = N(\chi), \tag{3}$$

where N is one of the two types in (2). Notice that Eq. (3) is equivalent to

$$D\Phi N = F \circ \Phi.$$

Our approach, which comes from ideas of Moser and DeLatte [4], consists on looking for a *remainder term* of the form

$$\text{(a)} \quad \widehat{B} = \begin{pmatrix} \xi\widehat{b}_1(\xi\eta, \mu\nu) \\ \eta\widehat{b}_1(\xi\eta, \mu\nu) \\ \mu\widehat{b}_2(\xi\eta, \mu\nu) \\ \nu\widehat{b}_2(\xi\eta, \mu\nu) \end{pmatrix} \quad \text{or} \quad \text{(b)} \quad \widehat{B} = \begin{pmatrix} \xi\widehat{b}_1(\xi\eta, \mu^2+\nu^2) \\ \eta\widehat{b}_1(\xi\eta, \mu^2+\nu^2) \\ \mu\widehat{b}_2(\xi\eta, \mu^2+\nu^2) \\ \nu\widehat{b}_2(\xi\eta, \mu^2+\nu^2) \end{pmatrix}, \tag{4}$$

depending if we are considering the saddle-focus or saddle-center case, respectively, satisfying $\widehat{b}_1(0,0) = \widehat{b}_2(0,0) = 0$ and such that the equality

$$D\Phi N + \widehat{B} = F \circ \Phi \tag{5}$$

holds. Note that (5) is equivalent to saying that the new system is of the form

$$\dot{\chi} = N(\chi) + (D\Phi(\chi))^{-1}\widehat{B}(\chi)$$

which is not, as a rule, a normal form. Thus, we will say that $X = \Phi(\chi)$ transforms system (1) into *pseudo-normal form* (ΨNF in short).

The interest of this construction lies in the following facts: first, it constitutes an extension of the BNF and, therefore, in the contexts where BNF *converges* they must coincide; second, this procedure is *convergent* in some situations where BNF does not apply and, thus, it translates the problem of the existence of a convergent normalizing transformation to the one of determining if some analytic scalar-valued functions $\widehat{b}_1$ and $\widehat{b}_2$ vanish. Finally, even in the case that these functions do not vanish, some interesting dynamical consequences can be derived from this pseudo-normal form.

Theorem 1 (Main Theorem). *Given a four-dimensional system*

$$\dot{X} = F(X) = \Lambda X + \widehat{F}(X), \tag{6}$$

analytic around the origin, where $\widehat{F}(X)$ denotes terms of order at least 2 in X, and with characteristic exponents $\{\pm\lambda_1, \pm\lambda_2\}$ equal to

- $\{\pm\lambda \pm \mathrm{i}\alpha\}$ *with $\lambda > 0$, $\alpha > 0$ (saddle-focus case), or*
- $\{\pm\lambda, \pm\mathrm{i}\alpha\}$ *with $\lambda > 0$, $\alpha > 0$ (saddle-center case),*

there exist an analytic transformation $X = \Phi(\chi) = \chi + \widehat{\Phi}(\chi)$ and analytic vector fields N, as in (2), and $\widehat{B}$, as in (4), in such a way that the equality

$$D\Phi N + \widehat{B} = F \circ \Phi$$

holds. Moreover, if system (6) is real analytic, Φ, N and B are also real analytic.

Section 2 is devoted to the proof of this theorem, which is constructive. It is based on a recurrent scheme which provides the coefficients of Φ, N and $\widehat{B}$, order by order. Moreover, a condition for determining the radius of convergence of these vector fields is provided in Eq. (64).

A first consequence of Theorem 1 is that for an initial *Hamiltonian* system, ΨNF becomes BNF.

Proposition H1. *System (6) is Hamiltonian in a neighborhood of the origin if and only if $\widehat{B}$ vanishes (and, therefore, ΨNF becomes BNF).*

The proof is given in Section 3. In the case that system (6) is a 2-degrees of freedom Hamiltonian, this proposition provides a new proof for the celebrated Moser's-Lyapunov theorem.

Corollary H2 (Lyapunov, Moser). *For an analytic Hamiltonian system around a saddle-focus or a saddle-center equilibrium*, BNF *is convergent*.

Some other consequences can be derived from a partial reading of Theorem 1. Namely, a linear center can be seen as a particular subsystem of the general saddle-center case. Indeed, if we write explicitly system (6) as

$$\begin{cases} \dot{x} = \lambda x + \widehat{f}_1(x,y,q,p), \\ \dot{y} = -\lambda y + \widehat{f}_2(x,y,q,p), \end{cases} \qquad \begin{cases} \dot{q} = \alpha p + \widehat{f}_3(x,y,q,p), \\ \dot{p} = -\alpha q + \widehat{f}_4(x,y,q,p) \end{cases} \tag{7}$$

for $\widehat{f}_1(0,0,q,p) = \widehat{f}_2(0,0,q,p) = 0$ and fix $x = y = 0$, we obtain the following planar system:

$$\begin{cases} \dot{q} = \alpha p + \widehat{f}_3(q,p), \\ \dot{p} = -\alpha q + \widehat{f}_4(q,p). \end{cases} \tag{8}$$

Here $\widehat{f}_j(q,p)$, $j = 3,4$ denote $\widehat{f}_j(0,0,q,p)$. This is the framework where the celebrated *center-focus problem* takes place. In this case Theorem 1 provides the existence of a transformation $(q,p) = \Phi(\mu,v)$ and vector fields $N(\mu,v)$ and $\widehat{B}(\mu,v)$, of the form

$$N = \begin{pmatrix} va(\mu^2+v^2) \\ -\mu a(\mu^2+v^2) \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} \mu\widehat{b}(\mu^2+v^2) \\ v\widehat{b}(\mu^2+v^2) \end{pmatrix}, \tag{9}$$

analytic in a neighborhood of the origin, with $a(0) = \alpha$, $\widehat{b}(0) = 0$, and satisfying $D\Phi N + \widehat{B} = F_c \circ \Phi$, where $F_c(p,q) = (\alpha p + \widehat{f}_3(q,p), -\alpha q + \widehat{f}_4(q,p))$. The following corollary is a reformulation of Proposition H1.

Corollary H3. *Assume $\widehat{f}_3, \widehat{f}_4$ analytic at the origin. Then, the following statements are equivalent*:

(i) *System* (8) *is* (*locally*) *Hamiltonian.*
(ii) *The origin is a center.*
(iii) *The function $\widehat{b}(\mu^2+v^2)$ in* (9) *provided by Theorem* 1 *vanishes identically.*

On the other hand, assuming $\widehat{f}_3 \equiv \widehat{f}_4 \equiv 0$ in system (7) (that is, the origin is a center in the (q,p)-variables), taking polar coordinates, scaling time if necessary and fixing an invariant cycle, we have a system of the form

$$\begin{cases} \dot{x} = \lambda x + \widehat{g}_1(x,y,\theta), \\ \dot{y} = -\lambda y + \widehat{g}_2(x,y,\theta), \\ \dot{\theta} = 1, \end{cases} \tag{10}$$

where $\gamma = \{x = y = 0\}$ is now a *hyperbolic periodic orbit* (of characteristic exponents $\pm\lambda$, $\lambda > 0$) and $\widehat{g}_1$, $\widehat{g}_2$ are analytic functions of x, y and θ. For such a system we have from Proposition H1 the following result.

Corollary H4 (Moser [13]). *Assume* (10) *is an analytic Hamiltonian system. Then, there exists a convergent transformation leading system* (10) *into* ΨNF *in a neighborhood of* γ *and this* ΨNF *coincides with the* BNF.

It is worth noticing that the original result due to Moser is also valid assuming only $\widehat{g}_1$ and $\widehat{g}_2$ to be $\mathscr{C}^1$ with respect to the angular variable θ. With a similar scheme to the one presented in this paper, Corollary H4 can also be proved under these weaker assumptions.

Up to this point, the results already presented follow from a suitable reading of Theorem 1 in a Hamiltonian framework. However, this is not the unique context where they can be applied. Namely, these results have a counterpart in the well known setting of the *reversible systems*.

We say that a system $\dot{X} = F(X)$ is $\mathfrak{G}$(time-)*reversible* (or simply, $\mathfrak{G}$-*reversible*) if it is invariant under $X \mapsto \mathfrak{G}(X)$ and a reversion in the direction of time $t \mapsto -t$, with $\mathfrak{G}$ being an involutory diffeomorphism, that is, $\mathfrak{G}^2 = \mathrm{id}$ and $\mathfrak{G} \neq \mathrm{id}$. From this definition, it turns out that F satisfies

$$\mathfrak{G}^* F = -F, \tag{11}$$

where $\mathfrak{G}^* F = (D\mathfrak{G})^{-1} F(\mathfrak{G})$. The diffeomorphism $\mathfrak{G}$ is commonly called a *reversing* involution of this system and is, in general, non linear. In this work we are dealing with analytic systems, so we will consider analytic involutions $\mathfrak{G}$. A set S which is invariant under the action of $\mathfrak{G}$ (that is, $\mathfrak{G}(S) \subseteq S$) is called $\mathfrak{G}$-*symmetric* or, simply, *symmetric* if there is no problem of misunderstanding. Since we are dealing with systems in a neighborhood of an equilibrium point or a periodic orbit, from now on we will assume always that these elements are symmetric with respect to the corresponding involution $\mathfrak{G}$.

Important examples of reversible systems are provided by the BNF (2). For instance, the BNF around a saddle-center equilibrium point (case (b) in (2)) is $\mathfrak{R}$-reversible, $\mathfrak{R}$ being the linear involution $(\xi, \eta, \mu, v) \mapsto (\eta, \xi, \mu, -v)$. Analogously, the BNF around a saddle-focus equilibrium point (case (a) in (2)) is reversible with respect to the linear involution $(\xi, \eta, \mu, v) \mapsto (\eta, \xi, v, \mu)$.

Proposition R1. *System* (6) *is reversible in a neighborhood of the origin if and only if* $\widehat{B}$ *vanishes* (*and*, *therefore*, ΨNF *becomes* BNF).

We recall that the Reversible Lyapunov Theorem was proven by Devaney [7] in both the smooth and the analytic case, using a geometrical approach. An alternative proof for this theorem is due to Vanderbauwhede [17] (see also [16,10], for an extension to families of analytic reversible vector fields).

The proof of this proposition is provided in Section 3. Notice that, in particular, it implies that locally Hamiltonian and locally reversible is the same around this equilibrium point. As in the Hamiltonian case, we have

Corollary R2. *Corollaries* H3 *and* H4 *also hold substituting Hamiltonian by reversible.*

From these results, it seems natural to look for a summarizing statement connecting both contexts, the Hamiltonian and the reversible. Indeed, we can summarize the previous statements in the following theorem.

Theorem 2. *Let us consider an analytic system*

$$\dot{X} = F(X) \tag{12}$$

and assume that one of the following three situations holds (*corresponding to dimensions* 2, 3 *and* 4, *respectively*),

(i) $X = (q, p) \in \mathbb{R}^2$ *and the origin is a linear center equilibrium point* (*like in system* (8)).
(ii) $X = (x, y, \theta) \in \mathbb{R}^2 \times \mathbb{T}$ *and* $\gamma = \{x = y = 0\}$ *is a hyperbolic periodic orbit* (*like in system* (10)).
(iii) $X = (x, y, q, p) \in \mathbb{R}^4$ *and the origin is a saddle-center or saddle-focus equilibrium point* (*like in system* (7)).

Then, *in a neighborhood of the corresponding critical element*, *the following statements are equivalent*

(i) *System* (12) *is Hamiltonian* (*with respect to some suitable* 2-*form* ω).
(ii) *System* (12) *is reversible* (*with respect to some suitable reversing involution* $\mathfrak{G}$).
(iii) *The analytic vector field* $\widehat{B}$ (*as in* (4)) *provided by Theorem* 1 *vanishes.*

This local duality around critical elements between Hamiltonian and reversible systems is quite common. As an example, see for instance [12], where it is proved this equivalence in the case of a nonsemisimple 1 : 1 resonance, which occurs when two pairs of purely imaginary eigenvalues of the linearized system collide. Nevertheless, there exist also counter examples of such equivalence. For instance, see the one given at [15], where it is given a class of area preserving mappings, with linear part the identity, which are not reversible.

Beyond the consequences provided by Theorem 1 in the Hamiltonian or reversible frameworks, this ΨNF-approach can be useful to find out isolated periodic orbits and related invariant manifolds in other situations. For instance, in [5] it is shown that for the center-focus problem (case (i) in Theorem 2) each zero of the analytic function $\widehat{b}$, defined in (9), gives rise to a limit cycle of system (8) close to the origin.

Now, consider system (12) with the origin being a saddle-center equilibrium point (case (iii) in Theorem 2). Let N and $\widehat{B}$, as in (2b), (4b), be the analytic vector fields provided by Theorem 1. Assume this system (12) is not locally Hamiltonian (neither reversible, therefore). Equivalently, functions $\widehat{b}_1$, $\widehat{b}_2$ in Eq. (4b) do not vanish simultaneously. Then the transformed system becomes of the form

$$\dot{\chi} = N(\chi) + (D\Phi(\chi))^{-1}\widehat{B}(\chi)$$

or, more precisely,

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\mu} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \xi a_1(\xi\eta, \mu^2+v^2) \\ -\eta a_1(\xi\eta, \mu^2+v^2) \\ v a_2(\xi\eta, \mu^2+v^2) \\ -\mu a_2(\xi\eta, \mu^2+v^2) \end{pmatrix} + (D\Phi(\chi))^{-1} \begin{pmatrix} \xi\widehat{b}_1(\xi\eta, \mu^2+v^2) \\ \eta\widehat{b}_1(\xi\eta, \mu^2+v^2) \\ \mu\widehat{b}_2(\xi\eta, \mu^2+v^2) \\ v\widehat{b}_2(\xi\eta, \mu^2+v^2) \end{pmatrix}. \tag{13}$$

Assume that $\widehat{b}_2$ does not vanish identically but there exists, at least, a non-zero value $I_* > 0$ satisfying $\widehat{b}_2(0, I_*) = 0$. If we take initial conditions $\xi^0 = \eta^0 = 0$ in (13) it follows that $\xi(t) = \eta(t) = 0\ \forall t$ and, therefore, $\mu^2 + v^2 = I_*$ becomes a limit cycle of the restricted system

$$\begin{cases} \dot{\mu} = v a_2(0, I_*), \\ \dot{v} = -\mu a_2(0, I_*), \end{cases} \tag{14}$$

where, for small enough values of I_*, we have $a_2(0, I_*) = \alpha + \mathrm{O}(I_*) \neq 0$. That is,

$$\Gamma_* = \{\mu^2 + v^2 = I_*\}$$

is a hyperbolic periodic orbit of system (14) with period $2\pi/a_2(0, I_*)$ and characteristic exponent $a_1(0, I_*) = \lambda + \mathrm{O}(I_*)$. Consequently,

$$\Gamma = \Phi(\Gamma_*) = \{\Phi(0, 0, \mu, v) : \mu^2 + v^2 = I_*\}$$

is a hyperbolic periodic orbit of system (12). It is also straightforward to parameterize the corresponding (local) stable and unstable invariant manifolds of Γ. Namely, there exists $\delta > 0$, given by the radius of convergence of the ΨNF, such that

$$W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma) = \{\Phi(0, \eta^0 e^{-ta_1(0,I_*)}, \mu, v) : |\eta^0| < \delta,\ \mu^2 + v^2 = I_*\},$$

$$W^{\mathrm{u}}_{\mathrm{loc}}(\Gamma) = \{\Phi(\xi^0 e^{ta_1(0,I_*)}, 0, \mu, v) : |\eta^0| < \delta,\ \mu^2 + v^2 = I_*\}. \tag{15}$$

We finish this introduction summarizing this result.

Corollary 3. *Consider system* (12) *where the origin is a saddle-center equilibrium point* (*case* (iii) *in Theorem* 2) *and let N and $\widehat{B}$, as in* (2b), (4b), *be the analytic vector fields provided by Theorem* 1. *Assume that the* (*analytic*) *function $I \mapsto \widehat{b}_2(0, I)$, defined in a neighborhood of the origin, does not vanish identically* (*so system* (12) *is neither Hamiltonian nor reversible*). *Thus, every positive zero of $\widehat{b}_2(0, *)$ gives rise to a* *hyperbolic periodic orbit of system* (12). *Moreover, parameterizations for the* (*local*) *stable and unstable invariant manifolds associated to this periodic orbit are given by* (15).

2. Proof of the main theorem

2.1. The formal solution: a first approach

It is worth noting that both cases, the origin being a saddle-focus or being a saddle-center, can be treated formally with the same argument. Moreover, we will deal first with the case of a *complex* ΨNF and will derive subsequently the case of a *real* ΨNF. Indeed, let us assume that we have complexified the original variables in such a way that the new (complex) matrix Λ is diagonal. Under this common approach, we will refer often to $\{\pm\lambda_1, \pm\lambda_2\}$ as the characteristic exponents of the origin, meaning $\{\pm\lambda\pm i\alpha\}$ in the first case and $\{\pm\lambda, \pm i\alpha\}$ in the second one, respectively, always with $\lambda, \alpha > 0$. Moreover, it is not difficult to check that with such unified notation the vector fields N and $\widehat{B}$ take the same form (2a) and (4a), respectively, in both cases. This will be their formal aspect along this proof if nothing against is explicitly said.

The sketch of the proof follows the standard pattern: first, we will look for a formal solution of equation

$$D\Phi N + \widehat{B} = F \circ \Phi \tag{16}$$

by means of a recurrent scheme, that will consist on two steps, an initial approach and a final refinement. Later on, it will be introduced a norm which will allow us to establish the convergence of the functions involved.

Thus, let us start with the first part. We recall that $\widehat{G}$ denotes that G is formed by formal power series beginning with terms of order at least 2. Now, since the linear part of $F(X) = \Lambda X + \widehat{F}(X)$ (or shorter, $F = \Lambda + \widehat{F}$) is in normal form, we have that the linear part of N is just Λ (notice that Λ represents also the complex matrix of eigenvalues $\pm\lambda_1, \pm\lambda_2$). Writing $\Phi = \mathrm{id} + \widehat{\Phi}$ and $N = \Lambda + \widehat{N}$, equation (16) becomes

$$D\widehat{\Phi} N - \Lambda\widehat{\Phi} = \widehat{F} \circ \Phi - \widehat{N} - \widehat{B}. \tag{17}$$

Assume that we already know $\widehat{\Phi}$, $\widehat{N}$ and $\widehat{B}$ up to some order K and let us see which difficulties involve the computation of the terms of order $K+1$ of $\widehat{\Phi}$. From Eq. (17) we realize that we only have to consider the terms up to order $K+1$ of equation

$$D\widehat{\Phi} N - \Lambda\widehat{\Phi} = \widehat{H}, \tag{18}$$

where $\widehat{H} = \widehat{F} \circ \Phi$ only contains terms up to order K of $\widehat{\Phi}$. The terms in $\widehat{N}$ and $\widehat{B}$ of order $K+1$ will be determined later. By direct computation, writing

$$\widehat{\Phi} = (\widehat{\phi}^{(1)}, \widehat{\phi}^{(2)}, \widehat{\phi}^{(3)}, \widehat{\phi}^{(4)}), \qquad \widehat{H} = (\widehat{h}^{(1)}, \widehat{h}^{(2)}, \widehat{h}^{(3)}, \widehat{h}^{(4)})$$

with

$$\widehat{\phi}^{(i)}(\xi, \eta, \mu, v) = \sum \phi^{(i)}_{jk\ell m} \xi^j \eta^k \mu^\ell v^m, \qquad \widehat{h}^{(i)}(\xi, \eta, \mu, v) = \sum h^{(i)}_{jk\ell m} \xi^j \eta^k \mu^\ell v^m,$$

for $i = 1, \dots, 4$, and using that N starts with $(\xi\lambda_1, -\eta\lambda_1, \mu\lambda_2, -v\lambda_2)$ the terms up to order $K+1$ of Eq. (18) come from the system,

$$(\xi \widehat{\phi}^{(i)}_\xi - \eta \widehat{\phi}^{(i)}_\eta) a_1(\xi\eta, \mu v) + \left(\mu \widehat{\phi}^{(i)}_\mu - v \widehat{\phi}^{(i)}_v \right) a_2(\xi\eta, \mu v) - \lambda_i^* \widehat{\phi}^{(i)} = \widehat{h}^{(i)},$$

for $i = 1, 2, \dots, 4$ and $\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*$ equal to $\lambda_1, -\lambda_1, \lambda_2$ and $-\lambda_2$, respectively. Therefore, the terms of order $K+1$ of $\widehat{\Phi}$ come from

$$\begin{aligned}
\phi^{(1)}_{jk\ell m} &= \frac{h^{(1)}_{jk\ell m}}{\lambda_1(j-k-1) + \lambda_2(\ell - m)} \quad \text{if } j \neq k+1 \text{ or } \ell \neq m, \\
\phi^{(2)}_{jk\ell m} &= \frac{h^{(2)}_{jk\ell m}}{\lambda_1(j-k+1) + \lambda_2(\ell - m)} \quad \text{if } k \neq j+1 \text{ or } \ell \neq m, \\
\phi^{(3)}_{jk\ell m} &= \frac{h^{(3)}_{jk\ell m}}{\lambda_1(j-k) + \lambda_2(\ell - m - 1)} \quad \text{if } j \neq k \text{ or } \ell \neq m+1, \\
\phi^{(4)}_{jk\ell m} &= \frac{h^{(4)}_{jk\ell m}}{\lambda_1(j-k) + \lambda_2(\ell - m + 1)} \quad \text{if } j \neq k \text{ or } m \neq \ell+1.
\end{aligned} \tag{19}$$

It is clear from these equations that terms of the form

$$\begin{pmatrix} \xi \sum \phi^{(1)}_{k+1,kmm} \ (\xi\eta)^k (\mu v)^m \\ \eta \sum \phi^{(2)}_{j,j+1,\ell\ell} \ (\xi\eta)^j (\mu v)^\ell \\ \mu \sum \phi^{(3)}_{kk,m+1,m} \ (\xi\eta)^k (\mu v)^m \\ v \sum \phi^{(4)}_{jj,\ell,\ell+1} \ (\xi\eta)^j (\mu v)^\ell \end{pmatrix} \tag{20}$$

cannot be determined and remain in principle arbitrary. In terms of simply linear algebra this amounts to say that the transformation Φ is completely determined once it has fixed its projection on a suitable vectorial subspace, called *resonant subspace*.

2.2. Definition of the projections

The type of coefficients appearing in expression (20) and the remarks above motivate the following definition.

Definition 4. Given a formal series $h(\xi,\eta,\mu,\nu)=\sum h_{jk\ell m}\xi^j\eta^k\mu^\ell\nu^m$, we define the projections

$$P_1 h := \xi \sum_{k\geqslant 0,\ m\geqslant 1} h_{k+1,kmm}(\xi\eta)^k(\mu\nu)^m,$$

$$P_2 h := \eta \sum_{j\geqslant 0,\ \ell\geqslant 1} h_{j,j+1,\ell\ell}(\xi\eta)^j(\mu\nu)^\ell,$$

$$P_3 h := \mu \sum_{k\geqslant 1,\ m\geqslant 0} h_{kk,m+1,m}(\xi\eta)^k(\mu\nu)^m,$$

$$P_4 h := \nu \sum_{j\geqslant 1,\ \ell\geqslant 0} h_{jj\ell,\ell+1}(\xi\eta)^j(\mu\nu)^\ell.$$

Moreover, if $H=(h^{(1)},h^{(2)},h^{(3)},h^{(4)})$ is a (formal) vector field we define

$$\mathscr{P}H := (P_1h^{(1)},P_2h^{(2)},P_3h^{(3)},P_4h^{(4)}),\quad \mathscr{R}H := H-\mathscr{P}H.$$

As it has been noticed before, $\mathscr{P}\widehat{\Phi}$ corresponds to the terms which remain arbitrary from the solution of Eq. (18). Moreover, vector fields N and $\widehat{B}$ are invariant under the action of $\mathscr{P}$. This property will be used in the solution of Eq. (17). In this sense, we have the following lemma, whose proof is omitted since it consists on straightforward computations.

Lemma 5. *Given $N=\Lambda+\widehat{N}$ of the form* (2a), *the operator $\mathscr{L}_N$ defined as*

$$\mathscr{L}_N\Psi := D\Psi N-\Lambda\Psi \tag{21}$$

satisfies the following properties:

(i) *$\mathscr{L}_N\Psi$ is linear with respect to Ψ and N, that is*

$$\mathscr{L}_N(\Psi+\Psi')=\mathscr{L}_N\Psi+\mathscr{L}_N\Psi',\qquad \mathscr{L}_{N+N'}\Psi=\mathscr{L}_N\Psi+\mathscr{L}_{N'}\Psi.$$

(ii) *$\mathscr{L}_N$ preserves order, that is, $\mathscr{L}_N\Psi$ and Ψ start with terms in (ξ,η,μ,ν) of the same order.*

(iii) *The projections $\mathscr{P}$ and $\mathscr{R}$ commute with $\mathscr{L}_N$, that is,*

$$\mathscr{P}(\mathscr{L}_N\Psi)=\mathscr{L}_N(\mathscr{P}\Psi),\qquad \mathscr{R}(\mathscr{L}_N\Psi)=\mathscr{L}_N(\mathscr{R}\Psi).$$

2.3. The recurrent scheme

Let us come back to the solution of Eq. (17). Having in mind the definition of the operator $\mathscr{L}_N$ it can be written as

$$\mathscr{L}_N\widehat{\Phi} = \widehat{F}\circ\Phi - \widehat{N} - \widehat{B}, \tag{22}$$

which is of type (18) provided we take $\widehat{H} = \widehat{F}\circ\Phi - \widehat{N} - \widehat{B}$. In a first approach to this kind of equations we have shown that they could be solved recurrently for those terms in $\Phi = \mathrm{id} + \widehat{\Phi}$ of type $\mathscr{R}\widehat{\Phi}$, remaining those of the form $\mathscr{P}\widehat{\Phi}$ arbitrary. This fact suggests the idea of splitting the transformation we are looking for, Φ, into $\mathrm{id} + \mathscr{P}\widehat{\Phi} + \mathscr{R}\widehat{\Phi}$, to determine $\mathscr{R}\widehat{\Phi}$ from Eq. (22) and to choose a suitable value for $\mathscr{P}\widehat{\Phi}$.

Remark 6. In Normal Form theory it is standard to set $\mathscr{P}\widehat{\Phi} = 0$ in order to simplify the computations. However, it could be useful to take advantage of this freedom in some concrete situations.

Applying $\mathscr{R}$ onto Eq. (22),

$$\mathscr{R}(\mathscr{L}_N\widehat{\Phi}) = \mathscr{R}(\widehat{F}(\Phi)) - \mathscr{R}\widehat{N} - \mathscr{R}\widehat{B},$$

using Lemma 5 and taking into account that $\mathscr{R}\widehat{N} = \mathscr{R}\widehat{B} = 0$ if $\widehat{N}$ and $\widehat{B}$ are assumed to be of the form (2a) and (4a), respectively, we obtain the equation

$$\mathscr{L}_N(\mathscr{R}\widehat{\Phi}) = \mathscr{R}(\widehat{F}(\Phi)). \tag{23}$$

On the other hand, applying now $\mathscr{P}$ onto (22), taking again into account Lemma 5, the fact that $\mathscr{P}\widehat{N} = \widehat{N}$, $\mathscr{P}\widehat{B} = \widehat{B}$ and choosing $\mathscr{P}\widehat{\Phi} \equiv 0$, it follows that

$$\widehat{N} + \widehat{B} = \mathscr{P}(\widehat{F}(\Phi)). \tag{24}$$

A usual way to deal with such kind of equations is to consider it as a fixed point problem. Thus, we can set $\mathscr{P}\widehat{\Phi} \equiv 0$, take initial values

$$\Phi^{(1)} = \mathrm{id}, \quad N^{(1)} = \Lambda, \quad \widehat{B}^{(1)} = 0 \tag{25}$$

and obtain, recurrently,

$$\Phi^{(K+1)} = \mathrm{id} + \mathscr{R}\widehat{\Phi}^{(K+1)}, \quad N^{(K+1)} = \Lambda + \widehat{N}^{(K+1)}, \quad \widehat{B}^{(K+1)} \tag{26}$$

from equations

$$\mathscr{L}_{N^{(K)}}(\mathscr{R}\widehat{\Phi}^{(K+1)}) = \mathscr{R}(\widehat{F}(\Phi^{(K)})), \tag{27}$$

$$\widehat{N}^{(K+1)} + \widehat{B}^{(K+1)} = \mathscr{P}(\widehat{F}(\Phi^{(K)})). \tag{28}$$

We will see now how these two equations can be solved formally.

2.3.1. Solution of a $\mathscr{L}_N(\boldsymbol{R}\widehat{\Psi}) = \mathscr{R}\widehat{H}$*-type equation*

Assuming that we know the coefficients of N and $\mathscr{R}\widehat{H}$ up to a given order K, the coefficients of $\mathscr{R}\widehat{\Psi}$ of the same order will be determined from

$$\mathscr{L}_N(\mathscr{R}\widehat{\Psi}) = \mathscr{R}\widehat{H}. \tag{29}$$

Indeed, writing

$$\mathscr{R}\widehat{\Psi} = (\widehat{\psi}_1, \widehat{\psi}_2, \widehat{\psi}_3, \widehat{\psi}_4), \qquad \mathscr{R}\widehat{H} = (\widehat{h}_1, \widehat{h}_2, \widehat{h}_3, \widehat{h}_4),$$

where

$$\widehat{\psi}_w(\xi, \eta, \mu, v) = \sum \psi^{(w)}_{jk\ell m} \xi^j \eta^k \mu^\ell v^m, \qquad \widehat{h}_w(\xi, \eta, \mu, v) = \sum \widehat{h}^{(w)}_{jk\ell m} \xi^j \eta^k \mu^\ell v^m$$

for $w = 1, \ldots, 4$, and taking into account that $N(\xi, \eta, \mu, v)$ has the form (2a), with $a_i(\xi\eta, \mu v) = \lambda_i + \widehat{a}_i(\xi\eta, \mu v)$, it follows that the left-hand side of (29) is equivalent to

$$\begin{pmatrix} ((j-k-1)\lambda_1 + (\ell - m)\lambda_2) + (\xi\widehat{\psi}_{1,\xi} - \eta\widehat{\psi}_{1,\eta})\widehat{a}_1 + (\mu\widehat{\psi}_{1,\mu} - v\widehat{\psi}_{1,v})\widehat{a}_2 \\ ((j-k+1)\lambda_1 + (\ell - m)\lambda_2) + (\xi\widehat{\psi}_{2,\xi} - \eta\widehat{\psi}_{2,\eta})\widehat{a}_1 + (\mu\widehat{\psi}_{2,\mu} - v\widehat{\psi}_{2,v})\widehat{a}_2 \\ ((j-k)\lambda_1 + (\ell - m - 1)\lambda_2) + (\xi\widehat{\psi}_{3,\xi} - \eta\widehat{\psi}_{3,\eta})\widehat{a}_1 + (\mu\widehat{\psi}_{3,\mu} - v\widehat{\psi}_{3,v})\widehat{a}_2 \\ ((j-k)\lambda_1 + (\ell - m + 1)\lambda_2) + (\xi\widehat{\psi}_{4,\xi} - \eta\widehat{\psi}_{4,\eta})\widehat{a}_1 + (\mu\widehat{\psi}_{4,\mu} - v\widehat{\psi}_{4,v})\widehat{a}_2 \end{pmatrix}.$$

We can refer to this vector field, in short, as

$$(L^{(1)}_N \widehat{\psi}_1, L^{(2)}_N \widehat{\psi}_2, L^{(3)}_N \widehat{\psi}_3, L^{(4)}_N \widehat{\psi}_4)$$

and write its components, in formal power series expansion, as

$$L^{(w)}_N \widehat{\psi}_w(\xi, \eta, \mu, v) = \sum_{j+k+\ell+m \geqslant 2} \tilde{g}^{(w)}_{jk\ell m}(\xi\eta, \mu v) \psi^{(w)}_{jk\ell m} \xi^j \eta^k \mu^\ell v^m, \tag{30}$$

being

$$\tilde{g}^{(w)}_{jk\ell m}(\xi\eta, \mu v) := \gamma^{(w)}_{jk\ell m}(\lambda) + (j-k)\widehat{a}_1(\xi\eta, \mu v) + (\ell - m)\widehat{a}_2(\xi\eta, \mu v),$$

with

$$\gamma^{(w)}_{jk\ell m}(\lambda) := \begin{cases} (j-k-1)\lambda_1 + (\ell - m)\lambda_2 & \text{if } w = 1, \\ (j-k+1)\lambda_1 + (\ell - m)\lambda_2 & \text{if } w = 2, \\ (j-k)\lambda_1 + (\ell - m - 1)\lambda_2 & \text{if } w = 3, \\ (j-k)\lambda_1 + (\ell - m + 1)\lambda_2 & \text{if } w = 4. \end{cases}$$

Notice, from Eq. (30), that $\mathscr{L}_N$ acts on $\mathscr{R}\widehat{\Psi}$ *multiplying* each coefficient $\psi_{jk\ell m}$ by a function of the products $\xi\eta$ and μv. To take advantage of this feature we will express our formal series expansions in a more convenient way which will highlight those terms of the form $(\xi\eta)^p$ and $(\mu\nu)^q$. A similar idea was suggested in [6]. In our case it works as follows. For any component $\widehat{\psi}_w$ of $\mathscr{R}\widehat{\Psi}$ we have

$$\widehat{\psi}_w(\xi,\eta,\mu,\nu) = \sum \psi^{(w)}_{jk\ell m}\xi^j\eta^k\mu^\ell\nu^m = \sum \psi^{(w)}_{jk\ell m}\xi^{j-k}(\xi\eta)^k\mu^{\ell-m}(\mu\nu)^m. \tag{31}$$

Defining $p = j-k$, $q = \ell - m$ and taking into account that $j+k+\ell+m \geqslant 2$, $p + k \geqslant 0$ and $q + m \geqslant 0$, this expansion is equivalent to

$$\sum_{p,q\in\mathbb{Z}} \psi^{(w)}_{pq}(\xi\eta,\mu\nu)\xi^p\mu^q, \tag{32}$$

where

$$\psi^{(w)}_{pq}(\xi\eta,\mu\nu) = \sum_{(k,m)\in Q_{pq}} \psi^{(w)}_{p+k,k,q+m,m}(\xi\eta)^k(\mu\nu)^m \tag{33}$$

and

$$Q_{pq} := \left\{(k,m)\in(\mathbb{N}\cup\{0\})^2 \colon \begin{matrix} k\geqslant\max\{0,-p\} \\ m\geqslant\max\{0,-q\}\end{matrix},\ k+m\geqslant 1-\frac{p+q}{2}\right\}.$$

In the same way, for $\mathscr{R}\widehat{H}$ we get

$$\widehat{h}_w(\xi,\eta,\mu,\nu) = \sum_{p,q\in\mathbb{Z}} h^{(w)}_{pq}(\xi\eta,\mu\nu)\xi^p\mu^q,$$

where

$$h^{(w)}_{pq}(\xi,\eta,\mu,\nu) = \sum_{(k,m)\in Q_{pq}} h^{(w)}_{p+k,k,q+m,m}(\xi\eta)^k(\mu\nu)^m. \tag{34}$$

With this notation formula (30) becomes

$$\sum_{p,q\in\mathbb{Z}} g^{(w)}_{pq}(\xi\eta,\mu\nu)\psi^{(w)}_{pq}(\xi\eta,\mu\nu)\xi^p\mu^q,$$

where now

$$g^{(w)}_{pq}(\xi\eta,\mu\nu) := \Gamma^{(w)}_{pq}(\lambda) + p\widehat{a}_1(\xi\eta,\mu\nu) + q\widehat{a}_2(\xi\eta,\mu\nu)$$

being

$$\Gamma^{(w)}_{pq}(\lambda) := \begin{cases} (p-1)\lambda_1 + q\lambda_2 & \text{if } w=1, \\ (p+1)\lambda_1 + q\lambda_2 & \text{if } w=2, \\ p\lambda_1 + (q-1)\lambda_2 & \text{if } w=3, \\ p\lambda_1 + (q+1)\lambda_2 & \text{if } w=4. \end{cases} \tag{35}$$

Thus, equality (29) gives rise to the equations

$$L_N^{(w)}\widehat{\psi}_w(\xi,\eta,\mu,v) = \widehat{h}_w(\xi,\eta,\mu,v)$$

or, in formal series expansions,

$$\sum_{p,q\in\mathbb{Z}} g_{pq}^{(w)}(\xi\eta,\mu v)\psi_{pq}^{(w)}(\xi\eta,\mu v)\,\xi^p\mu^q = \sum_{p,q\in\mathbb{Z}} h_{pq}^{(w)}(\xi\eta,\mu v)\xi^p\mu^q,$$

whose formal solution is given by

$$\widehat{\psi}_w(\xi,\eta,\mu,v) = \sum_{p,q\in\mathbb{Z}} \psi_{pq}^{(w)}(\xi\eta,\mu v)\,\xi^p\mu^q \tag{36}$$

with the functions $\psi_{pq}^{(w)}(\xi\eta,\mu v)$ coming from

$$\psi_{pq}^{(w)}(\xi\eta,\mu v) = \frac{h_{pq}^{(w)}(\xi\eta,\mu v)}{g_{pq}^{(w)}(\xi\eta,\mu v)} = \frac{h_{pq}^{(w)}(\xi\eta,\mu v)}{\Gamma_{pq}^{(w)}(\lambda) + p\widehat{a}_1(\xi\eta,\mu v) + q\widehat{a}_2(\xi\eta,\mu v)}, \tag{37}$$

for $w = 1,2,\dots,4$ and $p,q\in\mathbb{Z}$. With this notation coefficients with $p=\pm1$ and $q=0$ or $p=0$ and $q=\pm1$ are those belonging to the projection $\mathscr{P}\widehat{\boldsymbol{\Psi}}$.

2.3.2. Solution of a $\widehat{N}+\widehat{\boldsymbol{B}} = \boldsymbol{P}\widehat{\boldsymbol{H}}$-type equation

As it has been done for equations of type $\mathscr{L}_N(\mathscr{R}\widehat{\boldsymbol{\Psi}}) = \mathscr{R}\widehat{\boldsymbol{H}}$ we are going to prove that equation $\widehat{N}+\widehat{\boldsymbol{B}} = \mathscr{P}\widehat{\boldsymbol{H}}$ determines uniquely the coefficients of $\widehat{N}$ and $\widehat{\boldsymbol{B}}$ provided they are of type (2a) and (4a), respectively, and that $\widehat{\boldsymbol{H}}$ is known. Thus, writing

$$\mathscr{P}\widehat{\boldsymbol{H}} = (\xi\widehat{h}_1, \eta\widehat{h}_2, \mu\widehat{h}_3, v\widehat{h}_4), \tag{38}$$

where $\widehat{h}_w$ are functions of $\xi\eta$ and μv, for $w=1,2,\dots,4$, the solution of this equation is given explicitly by

$$\widehat{a}_1 = \tfrac{1}{2}(\widehat{h}_1 - \widehat{h}_2), \qquad \widehat{b}_1 = \tfrac{1}{2}(\widehat{h}_1 + \widehat{h}_2),$$
$$\widehat{a}_2 = \tfrac{1}{2}(\widehat{h}_3 - \widehat{h}_4), \qquad \widehat{b}_2 = \tfrac{1}{2}(\widehat{h}_3 + \widehat{h}_4). \tag{39}$$

2.4. The recurrent scheme: an improvement

One of the features of this procedure is that it provides a constructive (and, therefore, implementable on a computer) way to determine $\widehat{\boldsymbol{\Phi}}$, N and $\widehat{\boldsymbol{B}}$. To do it we need to define (and allocate memory for them) data vectors representing these vector fields. Unfortunately, the scheme above implies to handle (and to recompute) the *complete* vectors storing $\widehat{\boldsymbol{\Phi}}$, N and $\widehat{\boldsymbol{B}}$, at any step of the process. This makes it slow and not much efficient. In this sense it is easy to refine it by paying attention on the order of the solutions of Eqs. (27)–(28).

Before going on with this refinement, let us introduce some notation. We will denote $G = \mathcal{O}_{[K]}$ if G is a homogeneous polynomial in the spatial variables ξ, η, μ, v of order exactly K. Besides, we will write $G = \mathcal{O}_K$ if G contains only terms of order greater or equal than K in these variables and $G = \mathcal{O}_{\leqslant K}$ if all the terms in G are of order less or equal than K. Thus, we have

Lemma 7. *At any step* $K \geqslant 1$ *of process* (25)–(28), *the terms*

$$\Phi^{(K+1)} - \Phi^{(K)}, \quad N^{(K+1)} - N^{(K)}, \quad \widehat{B}^{(K+1)} - \widehat{B}^{(K)}$$

are all three $\mathcal{O}_{K+1}$.

It is not difficult to prove this result inductively, applying the properties in Lemma 5 and using the Taylor expansion of $\widehat{F}(\Phi)$.

An important consequence of this lemma is the reduction of the computational effort of the recurrent scheme: in the Kth step of our recurrent scheme the coefficients of order less or equal than K computed from the previous iteration will remain invariant. Therefore, from now onwards we will consider

$$\widehat{\Phi}^{(K+1)} = \mathcal{O}_{\leqslant K+1}, \quad N^{(K+1)} = \mathcal{O}_{\leqslant K+1}, \quad \widehat{B}^{(K+1)} = \mathcal{O}_{\leqslant K+1},$$

obtained from Eqs. (27)–(28) taken only up to order $K+1$

$$\{\mathscr{L}_{N^{(K)}}(\mathscr{R}\widehat{\Phi}^{(K+1)})\}_{\leqslant K+1} = \{\mathscr{R}(\widehat{F}(\Phi^{(K)}))\}_{\leqslant K+1}, \tag{40}$$

$$\widehat{N}^{(K+1)} + \widehat{B}^{(K+1)} = \{\mathscr{P}(\widehat{F}(\Phi^{(K)}))\}_{\leqslant K+1}. \tag{41}$$

In particular, we will denote

$$\begin{aligned} &\Phi^{(K+1)} = \Phi^{(K)} + \Delta\Phi^{(K)}, \quad N^{(K+1)} = N^{(K)} + \Delta N^{(K)}, \\ &\widehat{B}^{(K+1)} = \widehat{B}^{(K)} + \Delta\widehat{B}^{(K)}, \end{aligned} \tag{42}$$

where $\Delta\Phi^{(K)}$, $\Delta N^{(K)}$ and $\Delta\widehat{B}^{(K)}$ are $\mathcal{O}_{[K+1]}$. From a computational point of view, at any step K of this recurrent scheme it would be just necessary to compute these incremental terms. Besides, since $\widehat{N}^{(K)}$ and $\widehat{B}^{(K)}$ contain only terms of odd order, it follows that

$$\Delta N^{(2J-1)} = \Delta\widehat{B}^{(2J-1)} = 0, \quad J \geqslant 2. \tag{43}$$

2.5. Convergence of the recurrent scheme

2.5.1. Definition of the norm, estimates and technical lemmas

The domains we consider are those of type

$$\overline{\mathscr{D}_\sigma} = \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_j| \leqslant \sigma \quad j = 1, 2, \ldots, n\},$$

where $r > 0$ and $|\cdot|$ denotes the standard modulo. By an analytic function $f(z)$ on $\overline{\mathscr{D}_\sigma}$ we mean a function with Taylor expansion

$$f(z) = \sum_{\alpha \in (\mathbb{N} \cup \{0\})^n} f_\alpha z^\alpha \tag{44}$$

(absolutely) convergent for any $z \in \overline{\mathscr{D}_\sigma}$. We use the standard multi-index notation. Given a function f analytic on $\overline{\mathscr{D}_\sigma}$ we consider the following norms:

$$||f||_{\infty,\sigma} = \sup_{z \in \overline{\mathscr{D}_\sigma}} |f(z)|, \quad ||f||_{1,\sigma} = \sum_{|\alpha| \geqslant 0} |f_\alpha| \sigma^{|\alpha|},$$

the *supremum norm* and the 1-*norm*, respectively. For a vector field $F = (f_1, f_2, \ldots, f_n) : \overline{\mathscr{D}_\sigma} \subseteq \mathbb{C}^n \mapsto \mathbb{C}^n$ we define

$$||F||_{\infty,\sigma} = \sup_{i=1,\ldots,n} ||f_i||_{\infty,\sigma}, \qquad ||F||_{1,\sigma} = \frac{1}{n} \sum_{i=1,\ldots,n} ||f_i||_{1,\sigma} \tag{45}$$

and analogously if $F : \overline{\mathscr{D}_\sigma} \subseteq \mathbb{C}^n \mapsto \mathbb{M}_{n,n}(\mathbb{C}^n)$. The next lemma list some properties of these norms. We omit its proof since it is standard.

Lemma 8. *Let f be an analytic function on $\overline{\mathscr{D}_{\sigma_1}}$ satisfying that $f(0) = 0$ and assume $0 < \sigma_2 \leqslant \sigma_1$. Then, the following properties hold*:

(i) $||f||_{\infty,\sigma_2} \leqslant ||f||_{1,\sigma_2}$.

(ii) *Let $\Phi = (\phi_1, \phi_2, \ldots, \phi_n) : \overline{\mathscr{D}_{\sigma_2}} \subseteq \mathbb{C}^n \mapsto \mathbb{C}^n$ be analytic on $\overline{\mathscr{D}_{\sigma_2}}$ and satisfying that $||\Phi||_{\infty,\sigma_2} \leqslant \sigma_1$ (that is, $\Phi(\overline{\mathscr{D}_{\sigma_2}}) \subseteq \overline{\mathscr{D}_{\sigma_1}}$). Then we have*

$$||f \circ \Phi||_{1,\sigma_2} \leqslant ||f||_{1,\sigma_1}.$$

If $F = (f_1, \ldots, f_n)$ is an analytic vector field on $\overline{\mathscr{D}_{\sigma_1}}$ the same estimate holds for $||F \circ \Phi||_{1,\sigma_2}$.

(iii) *Let g be an analytic function on $\overline{\mathscr{D}_\sigma}$ satisfying that $|g(z)| \geqslant C \; \forall z \in \overline{\mathscr{D}_\sigma}$. Then, one has that*

$$\left|\left|\frac{1}{g}\right|\right|_{1,\sigma} \leqslant \frac{1}{C}.$$

(iv) *If* $G_{[K]} = \mathcal{O}_{[K]}$ *and* $H_{[L]} = \mathcal{O}_{[L]}$ *are homogeneous polynomials of orders* K *and* L, *respectively, with* $K \neq L$, *then*

$$||G_{[K]} + H_{[L]}||_{1,\sigma_2} = ||G_{[K]}||_{1,\sigma_2} + ||H_{[L]}||_{1,\sigma_2}.$$

From this point up to the end of this section we will prove some technical results which will be used during the proof of the convergence of the recurrent scheme introduced in Sections 2.3 and 2.4. In particular, next lemma provides a lower bound for $|q_1\lambda_1 + q_2\lambda_2|$ which works in both cases, when the equilibrium point is a *saddle-center* or a *saddle-focus* (whose characteristic exponents are given by $\{\pm\lambda, \mathrm{i}\alpha\}$ and $\{\pm\lambda\pm\mathrm{i}\alpha\}$, respectively).

Lemma 9. *Let us define*

$$\omega_\infty = \omega_\infty(\Lambda) := \min\{\lambda, \alpha\}, \tag{46}$$

where we assume $\lambda, \alpha > 0$. *Then, we have that*

$$|q_1\lambda_1 + q_2\lambda_2| \geqslant \left(\sqrt{q_1^2 + q_2^2}\right)\omega_\infty$$

for any $q_1, q_2 \in \mathbb{Z}$.

Proof. We proceed separately. Thus,

- *Saddle-center case*: As it has been mentioned above, we have $\lambda_1 = \lambda$ and $\lambda_2 = \mathrm{i}\alpha$ so

$$|q_1\lambda_1 + q_2\lambda_2| = \sqrt{q_1^2\lambda^2 + q_2^2\alpha^2} \geqslant \left(\sqrt{q_1^2 + q_2^2}\right)\min\{\lambda, \alpha\} = \left(\sqrt{q_1^2 + q_2^2}\right)\omega_\infty.$$

- *Saddle-focus case*: Now we have $\lambda_1 = \lambda + \mathrm{i}\alpha$ and $\lambda_2 = \lambda - \mathrm{i}\alpha$. Then,

$$|q_1\lambda_1 + q_2\lambda_2| = |(q_1 + q_2)\lambda + (q_1 - q_2)\mathrm{i}\alpha| = \sqrt{(q_1 + q_2)^2\lambda^2 + (q_1 - q_2)^2\alpha^2}.$$

If $q_1q_2 > 0$, using that $|q_1| + |q_2| \geqslant \sqrt{q_1^2 + q_2^2}$, one obtains that

$$\sqrt{(q_1 + q_2)^2\lambda^2 + (q_1 - q_2)^2\alpha^2} \geqslant \sqrt{(q_1 + q_2)^2\lambda^2} \geqslant \left(\sqrt{q_1^2 + q_2^2}\right)\omega_\infty.$$

On the other hand, if $q_1q_2 < 0$ then

$$\sqrt{(q_1 + q_2)^2\lambda^2 + (q_1 - q_2)^2\alpha^2} \geqslant \sqrt{(q_1 - q_2)^2\alpha^2} \geqslant \left(\sqrt{q_1^2 + q_2^2}\right)\omega_\infty. \qquad \square$$

Remark 10. In fact, ω_∞ constitutes a lower bound for the values ω_k introduced by Bruno in *condition* ω (see Section 1). Moreover, notice that, in the *saddle-center* case, one has that

$$\rho(\Lambda^{-1}) = \omega_\infty^{-1},$$

where $\rho(M)$ is the *spectral radius* of the matrix M, defined as the maximum of the modulus of their eigenvalues.

Now, we present a basic result which provides estimates for the vector fields $\mathscr{R}\widehat{\Psi}$, $\widehat{N}$ and $\widehat{B}$ that are solutions of the equations

$$\widehat{N} + \widehat{B} = \mathscr{P}\widehat{H}, \qquad \mathscr{L}_N(\mathscr{R}\widehat{\Psi}) = \mathscr{R}\widehat{H} \tag{47}$$

and whose formal approach has been derived in Sections 2.3.2 and 2.3.1, respectively.

Proposition 11. *Let us consider a vector field $\widehat{H}$ analytic on $\overline{\mathscr{D}_\sigma}$ and let $\mathscr{R}\widehat{\Psi}$ and $\widehat{N}$, $\widehat{B}$ (of the form* (2a) *and* (4a), *respectively) be the solutions of Eqs.* (47), *(formally) derived in Sections* 2.3.2 *and* 2.3.1. *Then, the following estimates hold.*

(i) *First, we have*

$$||\widehat{N}||_{1,\sigma},\ ||\widehat{B}||_{1,\sigma} \leqslant ||\mathscr{P}\widehat{H}||_{1,\sigma}.$$

(ii) *Moreover,*

$$||\mathscr{R}\widehat{\Psi}||_{1,\sigma} \leqslant \frac{||\mathscr{R}\widehat{H}||_{1,\sigma}}{\omega_\infty\left(1 - \frac{4}{\sigma\omega_\infty}||\mathscr{R}\widehat{H}||_{1,\sigma}\right)}$$

provided we assume that the following bound is satisfied:

$$||\mathscr{R}\widehat{H}||_{1,\sigma} < \frac{\sigma\omega_\infty}{4}. \tag{48}$$

Proof. (i) From (38) and (39) it follows that $||\widehat{N}||_{1,\sigma}$ and $||\widehat{B}||_{1,\sigma}$ are both bounded by $||\mathscr{P}\widehat{H}||_{1,\sigma}$.

(ii) The second equation in (47) was solved in Section 2.3.1. From there it follows that

$$\begin{aligned} ||\widehat{h}_w||_{1,\sigma} &= \sum_{j+k+\ell+m \geqslant 2} |h_{jk\ell m}|\sigma^{j+k+\ell+m} \\ &= \sum_{p,q\in\mathbb{Z}} \sum_{(k,m)\in Q_{pq}} |h^{(w)}_{p+k,k,q+m,m}|\sigma^{p+q+2(k+m)} \\ &= \sum_{p,q\in\mathbb{Z}} ||h^{(w)}_{pq}(\xi\eta,\mu\nu)\,\xi^p\mu^q||_{1,\sigma}. \end{aligned} \tag{49}$$

From that section we also know that the solution $\mathscr{R}\widehat{\Psi} = (\widehat{\psi}_1, \widehat{\psi}_2, \dots, \widehat{\psi}_4)$ is given, in terms of formal power series by (32) where $\psi_{pq}^{(w)}(\xi\eta, \mu v)$ are obtained from

$$\psi_{pq}^{(w)}(\xi\eta, \mu v) = \frac{h_{pq}^{(w)}(\xi\eta, \mu v)}{\Gamma_{pq}^{(w)}(\lambda) + p\widehat{a}_1(\xi\eta, \mu v) + q\widehat{a}_2(\xi\eta, \mu v)}, \tag{50}$$

for $w = 1, 2, \dots, 4$, $p, q \in \mathbb{Z}$ and the coefficients $\Gamma_{pq}^{(w)}(\lambda)$ as defined in (35). Notice that the functions $\psi_{pq}^{(w)}$ in (50) are rational functions of $\xi\eta$, μv. Therefore, expression (32) is not a standard representation in power series, that is, formula (34) does not apply to $\psi_{pq}^{(w)}$.

To estimate the 1-norm of $\mathscr{R}\widehat{\Psi}$ on $\overline{\mathscr{D}_\sigma}$ we have to bound their components. Taking into account Lemma 8(i) we have

$$\begin{aligned} ||\widehat{\psi}_w||_{1,\sigma} &\leqslant \sum_{p,q\in\mathbb{Z}} ||\psi_{pq}^{(w)}(\xi\eta, \mu v)\xi^p\mu^q||_{1,\sigma} \\ &= \sum_{p,q\in\mathbb{Z}} \left|\left| \frac{h_{pq}^{(w)}(\xi\eta, \mu v)\xi^p\mu^q}{\Gamma_{pq}^{(w)}(\lambda) + p\widehat{a}_1(\xi\eta, \mu v) + q\widehat{a}_2(\xi\eta, \mu v)} \right|\right|_{1,\sigma} \\ &\leqslant \sum_{p,q\in\mathbb{Z}} ||h_{pq}^{(w)}(\xi\eta, \mu v)\xi^p\mu^q||_{1,\sigma} \left|\left| \frac{1}{\Gamma_{pq}^{(w)}(\lambda) + p\widehat{a}_1(\xi\eta, \mu v) + q\widehat{a}_2(\xi\eta, \mu v)} \right|\right|_{1,\sigma}. \end{aligned} \tag{51}$$

Next lemma gives an upper bound for the second norm appearing in this formula (51).

Lemma 12. *Consider $\Gamma_{pq}^{(w)}(\lambda)$ as defined in* (35) *and $\widehat{a}_1(\xi\eta, \mu v)$, $\widehat{a}_2(\xi\eta, \mu v)$ coming from* (2a). *Then, for any $p, q \in \mathbb{Z}$ and $(\xi, \eta, \mu, v) \in \overline{\mathscr{D}_\sigma}$, we have that*

$$|\Gamma_{pq}^{(w)}(\lambda) + p\widehat{a}_1(\xi\eta, \mu v) + q\widehat{a}_2(\xi\eta, \mu v)| \geqslant \omega_\infty \left(1 - \frac{4}{\sigma\omega_\infty} ||\mathscr{R}\widehat{H}||_{1,\sigma}\right)$$

provided estimate (48) *is satisfied.*

Proof. We will distinguish two cases:

- If $|p| + |q| \geqslant 1$ it follows that

$$|\Gamma_{pq}^{(w)}(\lambda) + p\widehat{a}_1 + q\widehat{a}_2| \geqslant ||\Gamma_{pq}^{(w)}(\lambda)| - ||p\widehat{a}_1 + q\widehat{a}_2||_{\infty,\sigma}|. \tag{52}$$

From the definition of $\Gamma_{pq}^{(w)}$ in (35) and applying Lemma 9 it turns out that

$$|\Gamma_{pq}^{(w)}(\lambda)| \geqslant M_{pq}\omega_\infty,$$

where we define

$$M_{pq} := \min\left\{ \sqrt{(|p|-1)^2+q^2}, \sqrt{p^2+(|q|-1)^2} \right\}.$$

We recall that the terms $h_{pq}^{(w)}(\xi\eta,\mu\nu)$ with $|p|=1$ and $q=0$ or $p=0$ and $|q|=1$ vanish since they belong to the projection $\mathscr{P}\widehat{H}$ so, in particular, this implies that

$$M_{pq} \geqslant 1. \tag{53}$$

Moreover, it is clear that

$$|p|, |q| \leqslant 2M_{pq}. \tag{54}$$

Coming back to Eq. (52) we have that

$$\begin{aligned} &|\,|\Gamma_{pq}^{(w)}(\lambda)| - ||p\widehat{a}_1 + q\widehat{a}_2||_{\infty,\sigma}| \\ &\quad\geqslant M_{pq}\omega_\infty \left| 1 - \frac{1}{M_{pq}\omega_\infty} ||p\widehat{a}_1 + q\widehat{a}_2||_{\infty,\sigma} \right| \geqslant \omega_\infty \left(1 - \frac{4}{\sigma\omega_\infty} ||\mathscr{R}\widehat{H}||_{1,\sigma} \right), \end{aligned}$$

where it has been taken into account assumption (48) and, from (54), (39) and Lemma 8(i), the estimate

$$\begin{aligned} &\frac{1}{M_{pq}\omega_\infty} ||p\widehat{a}_1 + q\widehat{a}_2||_{\infty,\sigma} \\ &\quad\leqslant \frac{1}{\omega_\infty}\left(\frac{|p|}{M_{pq}} ||\widehat{a}_1||_{\infty,\sigma} + \frac{|q|}{M_{pq}} ||\widehat{a}_2||_{\infty,\sigma} \right) \leqslant \frac{2}{\omega_\infty}(||\widehat{a}_1||_{\infty,\sigma} + ||\widehat{a}_2||_{\infty,\sigma}) \\ &\quad= \frac{2}{\sigma\omega_\infty}(||\sigma\widehat{a}_1||_{\infty,\sigma} + ||\sigma\widehat{a}_2||_{\infty,\sigma}) \leqslant \frac{4}{\sigma\omega_\infty} ||\mathscr{R}\widehat{H}||_{\infty,\sigma} \leqslant \frac{4}{\sigma\omega_\infty} ||\mathscr{R}\widehat{H}||_{1,\sigma}. \end{aligned}$$

- If $p=q=0$ one has that

$$|\Gamma_{pq}^{(w)}(\lambda) + p\widehat{a}_1 + q\widehat{a}_2| = |\Gamma_{00}^{(w)}(\lambda)| \geqslant \omega_\infty, \tag{55}$$

and, in particular, assuming again (48),

$$|\Gamma_{00}^{(w)}(\lambda)| \geqslant \omega_\infty \geqslant \omega_\infty \left(1 - \frac{4}{\sigma\omega_\infty} ||\mathscr{R}\widehat{H}||_{1,\sigma} \right).$$

This concludes the proof of this lemma. $\square$

Since we are assuming that (48) holds, we can apply this lemma together with Lemma 8(iii) and, therefore, it follows that

$$\left\| \frac{1}{\Gamma_{pq}^{(w)}(\lambda) + p\widehat{a}_1(\xi\eta,\mu\nu) + q\widehat{a}_2(\xi\eta,\mu\nu)} \right\|_{1,\sigma} \leqslant \frac{1}{\omega_\infty \left(1 - \frac{4}{\sigma\omega_\infty} ||\mathscr{R}\widehat{H}||_{1,\sigma} \right)}.$$

Thus, estimate (51) jointly with (49) gives

$$\begin{aligned}||\widehat{\psi}_w||_{1,\sigma} &\leqslant \frac{1}{\omega_\infty\left(1-\frac{4}{\sigma\omega_\infty}||\mathscr{R}\widehat{H}||_{1,\sigma}\right)}\sum_{p,q\in\mathbb{Z}}||h_{pq}^{(w)}(\xi\eta,\mu\nu)\xi^p\mu^q||_{1,\sigma}\\ &= \frac{||\widehat{h}_w||_{1,\sigma}}{\omega_\infty\left(1-\frac{4}{\sigma\omega_\infty}||\mathscr{R}\widehat{H}||_{1,\sigma}\right)},\end{aligned}$$

for $w=1,2,\dots,4$. Finally, using (45), it turns out that

$$||\mathscr{R}\widehat{\Psi}||_{1,\sigma} \leqslant \frac{||\mathscr{R}\widehat{H}||_{1,\sigma}}{\omega_\infty\left(1-\frac{4}{\sigma\omega_\infty}\,||\mathscr{R}\widehat{H}||_{1,\sigma}\right)}. \qquad \square$$

2.5.2. Proof of the convergence

To ease the reading of this proof, let us recall briefly the problem we are dealing with. Let consider a system

$$\dot{X} = F(X) = \Lambda + \widehat{F}(X), \tag{56}$$

where F is analytic on a domain $\overline{\mathscr{D}_R}$ and having at $X=0$ a saddle-focus or saddle-center equilibrium point with characteristic exponents $\{\pm\lambda_1, \pm\lambda_2\}$ equal to $\{\pm\lambda\pm\mathrm{i}\alpha\}$ and $\{\pm\lambda,\ \pm\mathrm{i}\alpha\}$, respectively. As it has been seen at the beginning of Section 2.1, we can assume the matrix Λ to be written in (complex) diagonal form. This allows us to deal with both cases using a unified approach. We also recall that, again in Section 2.1, we introduced the notation Λ to denote both the matrix Λ and the vector field $\Lambda\mathrm{id}$. We will only use explicitly the second expression in cases of possible misunderstanding.

Our aim is the following: we are looking for an analytic transformation $X=\Phi(\chi)=\chi+\widehat{\Phi}(\chi)$ and analytic vector fields N and $\widehat{B}$ (that we can assume to be of the form (2a) and (4a), respectively) such that the equality

$$D\Phi N + \widehat{B} = F(\Phi) \tag{57}$$

is satisfied. We say in that case that Φ leads system (56) into ΨNF. To get such transformation and vector fields we have developed in Sections 2.3 and 2.4 the following *recurrent scheme* to whose convergence proof is devoted this section. Setting the following condition on $\widehat{\Phi}$,

$$\mathscr{P}\widehat{\Phi} \equiv 0, \tag{58}$$

we take initial values

$$\Phi^{(1)} = \mathrm{id}, \quad N^{(1)} = \Lambda, \quad \widehat{B}^{(1)} = 0 \tag{59}$$

and obtain, recurrently,

$$\Phi^{(K+1)} = \mathrm{id} + \mathscr{R}\widehat{\Phi}^{(K+1)}, \quad N^{(K+1)} = \Lambda + \widehat{N}^{(K+1)}, \quad \widehat{B}^{(K+1)} \tag{60}$$

with

$$\widehat{\Phi}^{(K+1)} = \mathcal{O}_{\leqslant K+1}, \quad \widehat{N}^{(K+1)} = \mathcal{O}_{\leqslant K+1}, \quad \widehat{B}^{(K+1)} = \mathcal{O}_{\leqslant K+1},$$

from equations

$$\{\mathscr{L}_{N^{(K)}}(\mathscr{R}\widehat{\Phi}^{(K+1)})\}_{\leqslant K+1} = \{\mathscr{R}(\widehat{F}(\Phi^{(K)}))\}_{\leqslant K+1}, \tag{61}$$

$$\widehat{N}^{(K+1)} + \widehat{B}^{(K+1)} = \{\mathscr{P}(\widehat{F}(\Phi^{(K)}))\}_{\leqslant K+1}. \tag{62}$$

Let us start with the proof. First, let us consider a positive constant $0<\gamma<1$ (in order to simplify the estimates, we can assume $\gamma \geqslant 1/2$, which is not restrictive). As it is commonly done in Normal Form Theory, we can scale our system by means of a change $X = \alpha Z$, where $\alpha>0$ is a constant to determine. Thus we have a new system

$$\dot{Z} = F_\alpha(Z) := \Lambda + \alpha^{-1}\widehat{F}(\alpha Z), \tag{63}$$

with F_α analytic on $\overline{\mathscr{D}_r}$, where $r := \alpha^{-1}R$. Let us consider a positive constant $0<\gamma<1$. In order to simplify the estimates, we can assume $\gamma \geqslant 1/2$, which is not restrictive. Then, since $\widehat{F}_\alpha$ starts with terms of order at least 2, we can choose α big enough (so r small enough) in such a way that the following estimate holds:

$$\|\widehat{F}\|_{1,r} \leqslant \left(\frac{(1-\gamma)\omega_\infty}{8}\right) r. \tag{64}$$

Calling again Z and F_α as X and F, respectively, we can assume our system (56) to be analytic on $\overline{\mathscr{D}_r}$ and satisfying (64). We are going to prove that the limit vector fields Φ, N and $\widehat{B}$ obtained from this recurrent scheme satisfy (57) and are analytic on $\overline{\mathscr{D}_{\gamma r}}$ (and therefore, reversing the scaling, on $\overline{\mathscr{D}_{\gamma R}}$).

We will itemize the proof in several parts: the first one will provide some estimates on the approximations provided by the recurrent scheme; in the second one, their convergence will be derived.

(i) Consider system (56) having F analytic on a domain $\overline{\mathscr{D}_r}$ and satisfying the assumption (64). Apply onto it the recurrent scheme (58)–(62) and consider the sequences

$$\{\|\Phi^{(K)}\|_{1,s}\}_K, \quad \{\|N^{(K)}\|_{1,s}\}_K, \quad \{\|\widehat{B}^{(K)}\|_{1,s}\}_K, \tag{65}$$

defined for $K \geqslant 1$ and being $s = \gamma r$. Then, the following properties are satisfied:

(a) They increase monotonically, that is,

$$||\Phi^{(K+1)}||_{1,s} \geqslant ||\Phi^{(K)}||_{1,s},$$

$$||N^{(K+1)}||_{1,s} \geqslant ||N^{(K)}||_{1,s}, \quad ||\widehat{B}^{(K+1)}||_{1,s} \geqslant ||\widehat{B}^{(K)}||_{1,s}.$$

(b) All these sequences are uniformly upper-bounded. Precisely, for all $K \geqslant 1$ we have that

$$||\Phi^{(K)}||_{1,s} \leqslant r \tag{66}$$

and that

$$||N^{(K)}||_{1,s}, ||\widehat{B}^{(K)}||_{1,s} \leqslant ||F||_{1,r}. \tag{67}$$

Let us prove these assertions.

(a) From Lemma 7, expressions (42), (43) and taking into account Lemma 8(iv), it turns out that

$$||\Phi^{(K+1)}||_{1,s} = ||\Phi^{(K)} + \mathscr{R}\Delta\widehat{\Phi}^{(K)}||_{1,s} = ||\Phi^{(K)}||_{1,s} + ||\mathscr{R}\Delta\Phi^{(K)}||_{1,s} \geqslant ||\Phi^{(K)}||_{1,s}.$$

The result for $||N^{(K+1)}||_{1,s}$ and $||\widehat{B}^{(K+1)}||_{1,s}$ can be derived analogously.

(b) To see it we proceed inductively. Thus, for $K = 1$ equation (61) becomes

$$\{\mathscr{L}_{N^{(1)}}(\mathscr{R}\widehat{\Phi}^{(2)})\}_{\leqslant 2} = \{\mathscr{R}(\widehat{F}(\Phi^{(1)}))\}_{\leqslant 2}.$$

Having in mind that $N^{(1)} = \Lambda$ (so $\widehat{N}^{(1)} = 0$), $\Phi^{(1)} = \mathrm{id}$ and definition (21) of the operator $\mathscr{L}$, this equation is equivalent to

$$D(\mathscr{R}\widehat{\Phi}^{(2)})\Lambda - \Lambda\mathscr{R}\widehat{\Phi}^{(2)} = \mathscr{R}F_{[2]}$$

and to

$$[\Lambda, \mathscr{R}\widehat{\Phi}^{(2)}] = F_{[2]},$$

where $[H, G] = (DG)H - (DH)G$ stands for the Lie bracket of the vector fields H and G. Now, from Proposition 11(ii), taking into account that $\widehat{a}_1^{(1)} = \widehat{a}_2^{(1)} = 0$ (the functions appearing in $\widehat{N}^{(1)}$) and using estimate (55) it follows that

$$||\mathscr{R}\widehat{\Phi}^{(2)}||_{1,s} \leqslant \frac{||F_{[2]}||_{1,s}}{\omega_\infty}$$

and, in particular,

$$\|\mathscr{R}\widehat{\Phi}^{(2)}\|_{1,s} \leqslant \frac{\|\widehat{F}\|_{1,r}}{\omega_\infty}. \tag{68}$$

Thus, applying Lemma 8(iv), the assumption $\mathscr{P}\widehat{\Phi} = 0$ and estimate (64), one obtains that

$$\|\Phi^{(2)}\|_{1,s} \leqslant s + \frac{\|\widehat{F}\|_{1,r}}{\omega_\infty} \leqslant \gamma r + \frac{1-\gamma}{8} r \leqslant \gamma r + (1-\gamma) r = r.$$

Concerning vector fields $N^{(2)}$ and $\widehat{B}^{(2)}$ we have that

$$N^{(2)} = N^{(1)} = \Lambda, \quad \widehat{B}^{(2)} = \widehat{B}^{(1)} = 0$$

and, therefore, estimate (67) satisfied. By induction hypothesis, let us now assume that bounds (66), (67) hold. We are going to show that they are also true for $K+1$. In fact, Eq. (61) is of type $\mathscr{L}_N(\mathscr{R}\widehat{\Psi}) = \mathscr{R}\widehat{H}$ provided we take

$$N = N^{(K)}, \quad \mathscr{R}\widehat{\Psi} = \mathscr{R}\widehat{\Phi}^{(K+1)}, \quad \mathscr{R}\widehat{H} = \mathscr{R}(\widehat{F}(\Phi^{(K)}))$$

and consider just terms up to order $K+1$. Setting $\sigma = s$ and taking into account estimate (64), the induction hypothesis and Lemma 8(i, ii) it follows that

$$\|\mathscr{R}\widehat{H}\|_{1,s} = \|\mathscr{R}(\widehat{F}(\Phi^{(K)}))\|_{1,s} \leqslant \|\widehat{F}\|_{1,r} \leqslant \left(\frac{(1-\gamma)\omega_\infty}{8}\right) r.$$

Using that $1/2 \leqslant \gamma < 1$ and that $s = \gamma r$, this estimate reads

$$\|\mathscr{R}\widehat{H}\|_{1,s} \leqslant \left(\frac{(1-\gamma)\omega_\infty}{8}\right) r \leqslant \frac{\gamma}{8}\omega_\infty r = \frac{s\omega_\infty}{8} < \frac{s\omega_\infty}{4},$$

which is assumption (48). Applying Proposition 11(ii) and that

$$\begin{aligned} 1 - \frac{4}{\gamma r \omega_\infty} \|\mathscr{R}(\widehat{F}(\Phi^{(K)}))\|_{1,\gamma r} &= 1 - \frac{4}{\gamma r \omega_\infty} \|\mathscr{R}\widehat{H}\|_{1,s} \\ &\geqslant 1 - \left(\frac{4}{s\omega_\infty}\right)\left(\frac{s\omega_\infty}{8}\right) = 1 - \frac{1}{2} = \frac{1}{2}, \end{aligned}$$

we obtain

$$\begin{aligned} \|\mathscr{R}\widehat{\Phi}^{(K+1)}\|_{1,\gamma r} &\leqslant \frac{\|\mathscr{R}(\widehat{F}(\Phi^{(K)}))\|_{1,\gamma r}}{\omega_\infty \left(1 - \frac{4}{\gamma r \omega_\infty}\|\mathscr{R}(\widehat{F}(\Phi^{(K)}))\|_{1,\gamma r}\right)} \\ &\leqslant \frac{\frac{(1-\gamma)\omega_\infty}{8} r}{\omega_\infty/2} = \frac{(1-\gamma) r}{4}. \end{aligned}$$

Finally, from Lemma 8(iv) one obtains that

$$||\Phi^{(K+1)}||_{1,s} = ||\Phi^{(K+1)}||_{1,\gamma r} = ||\mathrm{id}||_{1,\gamma r} + ||\mathscr{R}\widehat{\Phi}^{(K+1)}||_{1,\gamma r}$$
$$\leqslant \gamma r + \frac{(1-\gamma)r}{4} \leqslant \gamma r + (1-\gamma)r = r.$$

Concerning $N^{(K+1)}$ and $\widehat{B}^{(K+1)}$, having in mind the induction hypothesis $||\Phi^{(K)}||_{1,s} \leqslant r$, Eq. (62) and Section 2.3.2, one obtains that

$$||\widehat{N}^{(K+1)}||_{1,s} \leqslant ||\widehat{F}||_{1,r}, \quad ||\widehat{B}^{(K+1)}||_{1,s} \leqslant ||\widehat{F}||_{1,r} \leqslant ||F||_{1,r}.$$

Since $N^{(K+1)} = \Lambda + \widehat{N}^{(K+1)}$ and $F = \Lambda + \widehat{F}$ it turns out that

$$||N^{(K+1)}||_{1,s} \leqslant ||F||_{1,r}$$

which concludes the proof of (b).

(ii) At (i) it has been proved that the sequences

$$\{||\Phi^{(K)}||_{1,s}\}_K, \quad \{||N^{(K)}||_{1,s}\}_K, \quad \{||\widehat{B}^{(K)}||_{1,s}\}_K,$$

increase monotonically and are uniformly upper-bounded. Applying onto them the Ascoli-Arzelà theorem it follows that they admit convergent subsequences

$$\{||\Phi^{(K_J)}||_{1,s}\}_J, \quad \{||N^{(K_J)}||_{1,s}\}_J, \quad \{||\widehat{B}^{(K_J)}||_{1,s}\}_J.$$

Therefore, if we define a vector field Φ given by

$$\Phi(\chi) := \lim_{J\to\infty} \Phi^{(K_J)}(\chi)$$

for any $\chi \in \overline{\mathscr{D}_s}$, it follows that the limit

$$||\Phi||_{1,s} = \lim_{J\to\infty} ||\Phi^{(K_J)}||_{1,s}$$

exists and is finite. From Weierstrass theorem it follows that Φ is an analytic vector field on $\overline{\mathscr{D}_s} = \overline{\mathscr{D}_{\gamma r}}$. Moreover, since the recurrent scheme (58)–(62) and Lemma 7, provide vector fields $\Phi^{(K+1)}$ of the form

$$\Phi^{(K+1)} = \Phi^{(K)} + \mathscr{R}\Delta\widehat{\Phi}^{(K)}, \quad \mathscr{R}\Delta\widehat{\Phi}^{(K)} = \mathscr{O}_{[K+1]},$$

it can be derived that the subsequence $\{||\Phi^{(K_J)}||_{1,s}\}_J$ is, in fact, the complete sequence $\{||\Phi^{(K)}||_{1,s}\}_K$. In a similar way one obtains N and $\widehat{B}$, analytic vector fields on $\overline{\mathscr{D}_{\gamma r}}$ defined as

$$N := \lim_K N^{(K)}, \quad \widehat{B} := \lim_K \widehat{B}^{(K)}.$$

Together with Φ, they satisfy Eq. (57) and, therefore, they lead system (56) into ΨNF.

3. Proof of Propositions H1 and R1

3.1. *Proof of Proposition H1*

It is clear that if $\widehat{B} \equiv 0$ then ΨNF is just BNF so, let us consider the converse situation. To fix ideas, let us deal with a four-dimensional Hamiltonian system with the origin being a *saddle-center* equilibrium point. The *saddle-focus* case can be done in a similar way. Assume moreover that the center variables have been complexified (becoming complex conjugated). Applying Moser's Theorem [14], we know the existence of an analytic convergent transformation Ψ, close to the identity, leading it into BNF,

$$\begin{cases} \dot{\xi} = \xi a_1(\xi\eta, \mu v), \\ \dot{\eta} = -\eta a_1(\xi\eta, \mu v), \\ \dot{\mu} = \mu a_2(\xi\eta, \mu v), \\ \dot{v} = -v a_2(\xi\eta, \mu v) \end{cases} \tag{69}$$

with $a_1(\xi\eta, \mu v) = \lambda + \cdots$ and $a_2(\xi\eta, \mu v) = \mathrm{i}\alpha + \cdots$. It is clear that $\tilde{h}_1(\xi\eta) = \xi\eta\, a_1(\xi\eta, 0) = \lambda\xi\eta + \cdots$ and $\tilde{h}_2(\mu v) = \mu v a_2(0, \mu v) = \mathrm{i}\alpha\mu v + \cdots$ are independent *first integrals* of system (69) and, therefore,

$$h_1 = \tilde{h}_1 \circ \Psi^{-1} = \lambda xy + \cdots, \quad h_2 = \tilde{h}_2 \circ \Psi^{-1} = \mathrm{i}\alpha uv + \cdots$$

are independent first integrals of the original one. Let Φ be the convergent analytic transformation leading the initial system into ΨNF, that is, such that the new system is of the form

$$\dot{\chi} = N(\chi) + (D\Phi(\chi))^{-1}\widehat{B}(\chi), \tag{70}$$

where $\chi = (\xi, \eta, \mu, v)$ denotes now the ΨNF-variables. Since Φ starts with the identity and h_1, h_2 are independent first integrals of the original system, it follows that $\check{h}_1 = h_1 \circ \Phi$ and $\check{h}_2 = h_2 \circ \Phi$ are first integrals of (70) and, moreover, they begin with

$\lambda\xi\eta + \cdots$ and $\mathrm{i}\alpha\mu\nu + \cdots$, respectively. Indeed, they satisfy

$$D\check{h}_j(N + (D\Phi)^{-1}\widehat{B}) \equiv 0 \tag{71}$$

for $j = 1, 2$. Assume now that $\widehat{B} \neq 0$ so its minimal order terms are

$$\begin{pmatrix} \xi b_{rs}^{(1)}(\xi\eta)^r(\mu\nu)^s + \cdots \\ \eta b_{rs}^{(1)}(\xi\eta)^r(\mu\nu)^s + \cdots \\ \mu b_{r's'}^{(2)}(\xi\eta)^{r'}(\mu\nu)^{s'} + \cdots \\ \nu b_{r's'}^{(2)}(\xi\eta)^{r'}(\mu\nu)^{s'} + \cdots \end{pmatrix}$$

with $b_{rs}^{(1)} \neq 0$ or $b_{r's'}^{(2)} \neq 0$ (and $r + s$ not necessarily equal to $r' + s'$). Using that $\check{h}_1 = \lambda\xi\eta + \cdots$ and $(D\Phi)^{-1} = \mathrm{I} - (D\widehat{\Phi}) + \cdots$, the term of type $(\xi\eta)^{\ell}(\mu\nu)^m$ of minimal order corresponding to the left-hand side of Eq. (71), for $j = 1$, is given by

$$2\lambda b_{rs}^{(1)}(\xi\eta)^{r+1}(\mu\nu)^s + \cdots.$$

Since $\lambda \neq 0$ it implies that $b_{rs}^{(1)} = 0$. Applying the same argument to Eq. (71) with $j = 2$, and using that $\alpha \neq 0$, it follows that $b_{r's'}^{(2)} = 0$, which contradicts the assumption of $\widehat{B} \neq 0$. Consequently, $\widehat{B}$ vanishes.

3.2. *Proof of Proposition R1*

The problem of the convergence of the ΨNF (and BNF) around an equilibrium is certainly a local problem. In the reversible setting, this implies that both the linearized system and the reversing involution can be taken in suitable way. Namely, we have the following lemma whose proof is essentially contained in [16].

Lemma 13. *Let us consider a system $\dot{X} = F(X)$, analytic around the origin, a saddle-center or a saddle-focus equilibrium, and assume it is reversible with respect to an (in principle, nonlinear) involutory diffeomorphism $\mathfrak{G}$. Suppose that the origin is a fixed point of $\mathfrak{G}$. Then there exists an analytic change of variables $X \mapsto Z$, defined in a neighborhood of the origin, such that in the new coordinates the linearized system becomes $\dot{Z} = \Lambda$, with*

$$\Lambda = \mathrm{diag}(\lambda, -\lambda, \mathrm{i}\alpha, -\mathrm{i}\alpha)Z \tag{72}$$

or

$$\Lambda = \mathrm{diag}(\lambda + \mathrm{i}\alpha, \lambda - \mathrm{i}\alpha, -(\lambda + \mathrm{i}\alpha), -(\lambda - \mathrm{i}\alpha))Z, \tag{73}$$

depending if we are in the saddle-center or in saddle-focus case, respectively, and assuming $\lambda, \alpha > 0$. In such coordinates and in both cases, the symmetry $\mathfrak{G}$ can be taken of the form $Z \mapsto \mathfrak{R}Z$, where $\mathfrak{R}$ is the matrix associated to the *linear involution*

$$(x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_4, x_3). \tag{74}$$

Therefore, it is not restrictive to assume that our system is written, in a neighborhood of the origin, in the form

$$\dot{X} = F(X) = \Lambda + \widehat{F}(X), \tag{75}$$

with Λ as in (72) or (73) and that is (locally) reversible with respect to the linear involution $\mathfrak{R}$ defined in (74). Thus, the reversibility condition (11) reads

$$\mathfrak{R}F(\mathfrak{R}X) = -F(X). \tag{76}$$

Once we have set the linear framework, we present a property which characterizes those transformations that preserve a given linear reversibility.

Lemma 14. *Let Ψ be a diffeomorphism satisfying*

$$\mathfrak{R}\Psi(\mathfrak{R}\chi) = \Psi(\chi). \tag{77}$$

Then the transformation $X = \Psi(\chi)$ preserves the $\mathfrak{R}$-reversibility, that is, the new system

$$\dot{\chi} = G(\chi) := (\Psi^* F)(\chi)$$

is also $\mathfrak{R}$-reversible.

The proof of Proposition R1 is based on the following two points:

- Applying Theorem 1, there exist an analytic transformation $X = \Phi(\chi)$ and analytic vector fields $N(\chi)$, $\widehat{B}(\chi)$ leading the original system into ΨNF, provided the origin is a saddle-center or saddle-focus equilibrium point. That is, satisfying equality (57).
- The vector fields obtained from the recurrent scheme satisfy: (a) the transformation $X = \Phi(\chi)$ verifies relation (77), so it preserves $\mathfrak{R}$-reversibility; (b) N and $\widehat{B}$ are $\mathfrak{R}$-reversible. This last property will imply that $\widehat{B}$ has to vanish and, therefore, ΨNF will become BNF.

Lemma 15. *Let us consider an $\mathfrak{R}$-reversible system* (75), *analytic on a neighborhood of the origin, a saddle-center or a saddle-focus equilibrium point. Let us take $\Phi^{(K)}$, $N^{(K)}$ and $\widehat{B}^{(K)}$, the vector fields provided by the ΨNF-recurrent scheme* (58)–(62). *Then, the following assertions hold*:

(i) *For any $K \geqslant 1$, the vector field $\Phi^{(K)}$ satisfies* (77) *and the vector fields $N^{(K)}$ and $\widehat{B}^{(K)}$ are $\mathfrak{R}$-reversible, that is,*

$$\mathfrak{R}\Phi^{(K)}(\mathfrak{R}\chi) = \Phi^{(K)}(\chi),$$

and

$$\mathfrak{R}N^{(K)}(\mathfrak{R}\chi) = -N^{(K)}(\chi), \quad \mathfrak{R}\widehat{B}^{(K)}(\mathfrak{R}\chi) = -\widehat{B}^{(K)}(\chi).$$

(ii) *The vector fields Φ, N and $\widehat{B}$ provided by Theorem 1 and defined as*

$$\Phi = \lim_{K\to\infty} \Phi^{(K)}, \quad N = \lim_{K\to\infty} N^{(K)}, \quad \widehat{B} = \lim_{K\to\infty} \widehat{B}^{(K)},$$

verify the same properties as $\Phi^{(K)}$, $N^{(K)}$ and $\widehat{B}^{(K)}$ given above, respectively.

(iii) *Consequently, since $\widehat{B}$ is $\mathfrak{R}$-reversible it must vanish.*

Remark 16. Like in the Hamiltonian case, Φ is not completely determined. Namely, any choice for $\mathscr{P}\widehat{\Phi}$ being convergent and satisfying (77) gives rise to a different transformation Φ.

Proof. It is based in some statements that we list and whose proof can be obtained straightforwardly. Namely,

(a) If a vector field H is $\mathfrak{R}$-reversible then its projections $\mathscr{P}H$ and $\mathscr{R}H$ are also $\mathfrak{R}$-reversible.

(b) Let $\mathscr{R}\widehat{\Psi}$ be the solution of an equation of type (29). Then, if $\mathscr{R}\widehat{H}$ is $\mathfrak{R}$-reversible it follows that $\mathscr{R}\widehat{\Psi}$ satisfies (77) and, therefore, the transformation $X = \chi + \mathscr{R}\widehat{\Psi}(\chi)$ preserves the $\mathfrak{R}$-reversibility.

To see it, let us denote $\mathscr{R}\widehat{\Psi} = (\widehat{\psi}_1, \widehat{\psi}_2, \widehat{\psi}_3, \widehat{\psi}_4)$, where

$$\widehat{\psi}_w(\xi, \eta, \mu, v) = \sum_{j+k+\ell+m\geqslant 2} \psi^{(w)}_{jk\ell m} \tag{78}$$

for $w = 1, 2, \ldots, 4$. It is not difficult to check that if $\mathscr{R}\widehat{\Psi}$ satisfies Eq. (29), whose explicit solution is given in Section 2.3.1, then the coefficients in (78) verify that

$$\psi^{(2)}_{jk\ell m} = \psi^{(1)}_{kjm\ell}, \quad \psi^{(4)}_{jk\ell m} = \psi^{(3)}_{kjm\ell}$$

and, consequently, $\mathscr{R}\widehat{\Psi}$ satisfies relation (77).

(c) If H is $\mathfrak{R}$-reversible then $H_{\leqslant K}$ (constituted by its terms of order less or equal than K) is also $\mathfrak{R}$-reversible, for any $K \geqslant 1$.

We are now in conditions of proving assertions (i)–(iii).

(i) From its form, it is clear that $\widehat{N}^{(K)}$ and $N^{(K)}$ are $\mathfrak{R}$-reversible vector fields, for any $K \geqslant 1$. Now, we are going to prove that $\Phi^{(K)}$ verifies condition (77) and $\widehat{B}^{(K)}$ is $\mathfrak{R}$-reversible using an inductive argument.

For $K=2$ (the case $K=1$ is trivial) we have that

$$\{\mathscr{L}_{N^{(1)}}(\mathscr{R}\widehat{\Phi}^{(2)})\}_{\leqslant 2} = \{\mathscr{R}(\widehat{F}(\Phi^{(1)}))\}_{\leqslant 2}$$

or, simplifying,

$$\mathscr{L}_A(\mathscr{R}\widehat{\Phi}^{(2)}) = F_{[2]}.$$

Applying properties (c) and (b) above one obtains that $\mathscr{R}\widehat{\Phi}^{(2)}$ preserves $\mathfrak{R}$-reversibility. On the other hand, $\widehat{B}^{(2)}=0$ so it is trivially a $\mathfrak{R}$-reversible vector field. Assume now, as *induction hypotheses*, that for a given $K\geqslant 1$,

- $\Phi^{(K)} = \mathrm{id} + \mathscr{R}\widehat{\Phi}^{(K)}$ satisfies (77) (so it preserves $\mathfrak{R}$-reversibility),
- $\widehat{B}^{(K)}$ is a $\mathfrak{R}$-reversible vector field.

Applying properties (a), (c) and (b) on Eq. (61) it follows that $\mathscr{R}\widehat{\Phi}^{(K+1)}$ and, therefore, $\Phi^{(K+1)} = \mathrm{id} + \mathscr{R}\widehat{\Phi}^{(K+1)}$ preserve $\mathfrak{R}$-reversibility. Moreover, from Eq. (62) we have

$$\widehat{B}^{(K+1)} = \{\mathscr{P}(\widehat{F}(\Phi^{(K)}))\}_{\leqslant K+1} - \widehat{N}^{(K+1)}.$$

Thus, since $\Phi^{(K)}$ preserves $\mathfrak{R}$-reversibility, $\widehat{N}^{(K+1)}$ is $\mathfrak{R}$-reversible and taking into account properties (a), (c), it turns out that $\widehat{B}^{(K+1)}$ is also $\mathfrak{R}$-reversible.

(ii) It follows from (i) letting K tend to infinity and applying the (analytic) convergence of $\Phi^{(K)}$, $N^{(K)}$ and $\widehat{B}^{(K)}$.

(iii) From the $\mathfrak{R}$-reversibility of $\widehat{B}$,

$$\mathfrak{R}\widehat{B}(\mathfrak{R}\chi) = -\widehat{B}(\chi),$$

it turns out that

$$\widehat{b}_1(\xi\eta,\mu\nu) = -\widehat{b}_1(\xi\eta,\mu\nu) \quad \text{and} \quad \widehat{b}_2(\xi\eta,\mu\nu) = -\widehat{b}_2(\xi\eta,\mu\nu),$$

so $\widehat{b}_1(\xi\eta,\mu\nu) = \widehat{b}_2(\xi\eta,\mu\nu) = 0$ and the lemma is proved. $\square$

From this lemma the proof of Proposition R1 follows straightforwardly. The transformation Φ preserves $\mathfrak{R}$-reversibility, the vector field N is $\mathfrak{R}$-reversible and $\widehat{B}=0$ so, in fact, the ΨNF is nothing else but the BNF.

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