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Some words about the application of Tchebycheff systems to Weak Hilbert's 16th Problem

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This paper is dedicated to Jairo Antonio Charris.

Abstract. In this talk we will try to introduce (in a very naïve way) the so-called *Weak Hilbert's Problem*, posed by Arnol'd in 1977, its relation with the original Hilbert's 16th Problem and how Tchebycheff systems have been applied to approach them.

Contents

1. Hilbert's 16th and Weak Hilbert's 16th Problems

The aim of this lecture is to introduce the reader to the so-called *Weak Hilbert's 16th Problem* and to show how Tchebycheff systems can be used to give some approximative and partial solution to some concrete systems. It is addressed to a general audience with no expertise, in principle, on this topic and almost all the results contained are known. Thus we will spend an important part of this notes on introducing the problem, listing some of the main references and showing properties of the systems we will deal with but without entering into details. For more information we refer the reader to some references like, for instance, [**12, 14**] - which have been our main sources preparing this short lecture- , but the amount of papers written on this topic is huge as well as the number of researchers involved (Arnol'd, Bautin, Chen, Chicone, Christopher, Chow, Dulac, Dumortier,

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Écalle, Françoise, Gavrilov, Gasull, Iliev, Ilyashenko, Li, Llibre, Lloyd, Mardesic, Petrov, Poincaré, Pontryagin, Roussarie, Rousseau, Schlomiuk, Yakovenko, Zhang, Zoladek, ... and many others that we have not included in this short list (we apologize for that)). Let us now to introduce the framework of this lecture.

One of most exciting problems contained in Hilbert's celebrated list (see [**10**]), devoted to the so-called Problems of the XXth Century, is 16th Problem whose first part reads, in original version, approximatively like this: *Which is the number and relative position of Poincaré limit cycles (isolated periodic orbits) that can have a polynomial differential equation*

$$\frac{dy}{dx} = \frac{P_n(x,y)}{Q_n(x,y)}, \tag{1.1}$$

where P_n, Q_n are polynomials of degree n ?

Notice that the problem is trivial for $n = 1$ since a linear system cannot have limit cycles, so we will assume always that $n \geq 2$. Following [**12**], this problem admits at least three different specifications, which are

(1) *Individual finiteness problem:* Given a polynomial differential equation (1.1), with $deg\, P_n,\ Q_n \leq n$, to prove that it has only a finite number of limit cycles.

    Dulac claimed in 1923 that he had solved it but the proof he gave was erroneous. Much later, Ilyashenko in 1991 [**11**] and Écalle in 1992 [**6**] gave two different proofs of Dulac's Theorem.
(2) *Existential Hilbert Problem:* To prove that, for any $n \in \mathbb{N}$, the number of limit cycles of all polynomial differential equations (1.1) of degree less or equal than n is uniformly bounded. This uniform bound is denoted by $H(n)$ so this problem can be rewritten as to prove that $H(n) < \infty$ for any $n \in \mathbb{N}$.
(3) *Constructive Hilbert Problem:* To give an upper bound for $H(n)$ or to suggest an algorithm to get it.

It is important to stress that, of these three *problems* only the first one (the weakest) has been solved. A remarkable number of papers has been published concerning such problems providing partial solutions but no proof has been obtained for the general statements.

Regarding the second part of Hilbert's problem, the one concerning the configuration of limit cycles, we want just to mention the paper of Llibre and Rodríguez [**17**], where a general result was obtained.

Let us now introduce a weaker version on Hilbert's 16th Problem. It corresponds to a Problem posed by Arnol'd in 1977 and that can be stated as follows: Let $H = H(x, y)$ a polynomial in x, y of degree $m \geq 2$ and assume that the level curves $\gamma_h \subset \{(x, y)|\ H(x, y) = h\}$ form a continuous family of ovals $\{\gamma_h\}$ for $h_1 < h < h_2$. Consider now a polynomial 1-form $\omega = f(x, y)\, dy - g(x, y)\, dx$ with f, g polynomials satisfying that $\max\{deg(f), deg(g)\} = n \geq 2$. Then, the problem consists on, for fixed integer values m and n, finding the maximum number $Z(m, n)$ of isolated zeroes of the *Abelian integrals*

$$I(h) = \oint_{\gamma_h} \omega.$$

We recall that an Abelian integral is the integral of a rational 1-form along an algebraic oval. Following [**14**], we will show that this problem of counting the number of zeroes of an Abelian integral is closely related to Hilbert's 16th Problem. Let us see it.

Consider $H(x,y)$ a polynomial of degree $m \geq 2$ and the corresponding Hamiltonian vector field

$$\begin{cases} \dot{x} = -H_y(x,y) \\ \dot{y} = H_x(x,y) \end{cases}$$

where $\dot{}$ denotes d/dt and H_u stands for the partial derivative $\partial H/\partial u$, for $u = x, y$. Let us introduce ε an small parameter and consider the following perturbation of the previous system

$$\begin{cases} \dot{x} = -H_y(x,y) + \varepsilon f(x,y) \\ \dot{y} = H_x(x,y) + \varepsilon g(x,y). \end{cases} \tag{1.2}$$

For the unperturbed system ($\varepsilon = 0$), let us suppose that the origin is a center equilibrium point and that, therefore, for energies $h_1 < h < h_2$ there exists a family of ovals $\{\gamma_h\}$ varying continuously with h and filling up an *annulus* around it for $h \in (h_1, h_2)$. The corresponding Abelian integral reads in this case as

$$I(h) = \oint_{\gamma_h} f(x,y)\,dy - g(x,y)\,dx. \tag{1.3}$$

It is known that when we perturb such a system, i.e. take $\varepsilon \neq 0$, some of the initial periodic orbits γ_h (slightly deformed) will persist and will remain isolated. That is, they will become limit cycles of the unperturbed system. Indeed, the perturbation of a system having a linear center at the origin (that is, eigenvalues of its differential at the origin are purely imaginary conjugated complex) is a classical way to produce limit cycles.

A natural question which arises is the following: Is there any value $h \in (h_1, h_2)$ and some (isolated) periodic orbits Γ_ε of the perturbed system such that Γ_ε tend to γ_h as $\varepsilon \to 0$? How many of such Γ_ε can we have for the same h? If this happens we will say that Γ_ε *bifurcates* from γ_h.

The usual way to approach to this problem is to consider Σ, a transversal section (a segment in this case) to the periodic orbit γ_h, and to choose H to parameterize such a section Σ. We can denote by $\gamma_\varepsilon(h)$ the orbit of the perturbed system (1.2), close to γ_h provided ε small enough, with starting point h on Σ. We define $P(h,\varepsilon)$ the next intersection of $\gamma_\varepsilon(h)$ with Σ (see Fig. 1) and consider

$$d(h,\varepsilon) = P(h,\varepsilon) - h$$

the so-called *displacement function*. Notice that (isolated) zeroes of $d(h,\varepsilon)$ correspond to limit cycles of system (1.2). The following Theorem states a relationship between this displacement function and the Abelian integral (1.3).

THEOREM 1.1 (Poincaré-Pontryagin [**21, 22**]). *Under the hypotheses above one has that*

$$d(h,\varepsilon) = \varepsilon I(h) + \varepsilon^2 \phi(h,\varepsilon) \qquad \text{as } \varepsilon \to 0,$$

$\phi(h,\varepsilon)$ *being analytic and uniformly bounded in a compact region containing* $(h,0)$ *for* $h_1 < h < h_2$.

More precisely, next result provides a direct relation between zeroes of the Abelian integral $I(h)$ and existence of limit cycles.

THEOREM 1.2. *Provided the Abelian integral* $I(h)$ *in* (1.3) *is not identically zero for* $h_1 < h < h_2$*, the following statements hold:*

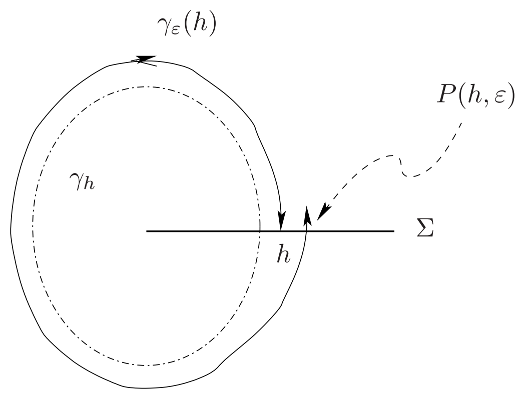

FIGURE 1. Displacement function

- *If system* (1.2) *has a limit cycle bifurcating from* γ_{h^*} *then* $I(h^*) = 0$.
- *If there exists* $h_1 < h^* < h_2$ *such that* $I(h^*) = 0$ *and* $I'(h^*) \neq 0$ *then system* (1.2) *has a unique limit cycle bifurcating from* γ_{h^*}. *Moreover, this limit cycle is hyperbolic.*
- *If* h^* *is a zero of order* k *of* $I(h)$, *i.e.* $I(h^*) = I'(h^*) = \ldots = I^{(k-1)}(h^*) = 0$ *and* $I^{(k)}(h^*) \neq 0$, *then system* (1.2) *has at most* k *limit cycles which bifurcate from* γ_{h^*} *(taking into account the multiplicity of the limit cycles).*
- *Finally, the total number (taking into account multiplicity) of limit cycles bifurcating from the* period annulus $\displaystyle\bigcup_{h_1<h<h_2} \gamma_h$ *associated to system* (1.2) *is bounded by the maximum number of isolated zeroes (counting again multiplicity) of the corresponding Abelian integral* $I(h)$ *for* $h_1 < h < h_2$.

A classical (and simple) example of application of this theorem is the following one.

Example: Let us consider the well known Van der Pol equation

$$\ddot{x} + \varepsilon \left(x^2 - 1\right) \dot{x} + x = 0$$

or, equivalently

$$\left\{ \begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & -x + \varepsilon \left(1 - x^2\right) y \end{array} \right. . \tag{1.4}$$

The unperturbed system ($\varepsilon = 0$) is Hamiltonian, with Hamilton function $H(x, y) = x^2 + y^2$ and the origin as an elliptic equilibrium point. It presents the following family of ovals $\{\gamma_h\}_h$ defined through

$$\gamma_h = \left\{(x, y) |\; x^2 + y^2 = h\right\}$$

defined for $h > 0$. It is straightforward to check, using polar coordinates, that in this case

$$I(h) = -\oint_{\gamma_h} (1 - x^2) y \, dx = \int_0^{2\pi} \left(1 - h^2 \cos^2\theta\right) h^2 \left(-sin^2\theta\right) \, d\theta = \pi h^2 \left(\frac{h^2}{4} - 1\right).$$

Notice that there is only one positive zero of $I(h)$ (the case $h = 0$ has not to be considered since it corresponds to the singularity) is $h = 2$. It is not difficult to check that $I'(2) = 4\pi \neq 0$ so, applying the previous theorem, we can deduce the existence of a unique limit cycle of system (1.4) for small enough values of ε. Moreover this limit cycle is hyperbolic.

Even though there exists an strong relation between the problem of counting the maximum number of zeroes of an Abelian integral $I(h)$ and the problem concerning the number of limit cycles appearing (locally) around an equilibrium, a recent result by Dumortier, Panazzolo and Roussarie [**5**] (pointed out to us by the referee) has shown that this relationship is not completely exact.

2. Estimating the number of zeroes of an Abelian integral

There exist different well known methods to approach the problem of counting (or giving bounds) the number of zeroes that a given Abelian integral $I(h)$ can have in its domain of definition. Among them, the most common are those based on the Picard-Fuchs equation, on the so-called Argument Principle and on the Averaging equation. The first one consists on seeking for and studying the differential equation satisfied by the *basis* of functions forming $I(h)$, that is, those functions $I_1(h), I_2(h), \ldots, I_n(h)$ such that

$$I(h) = p_1(h)I_1(h) + p_2(h)I_2(h) + \cdots + p_n(h)I_n(h)$$

with $p_1(h), p_2(h), \ldots, p_n(h)$ polynomials in h. The second one type of methods deals with Complex Analysis skills, like the Argument Principle itself and Rouché's Theorem, which allow us to estimate the number of zeroes that an analytic extension of $I(h)$ has in a suitable complex domain. Finally, the third one consists on deriving properties (if possible) of the original system from those of the system obtained after one or more steps of averaging.

Another type of methods, those we want to present with some more (non really much more) detail, are those based in the so-called Tchebycheff systems. In a few words, we would like to find functions $J_1(h), J_2(h), \ldots, J_N(h)$ such that the Abelian integral $I(h)$ associated to system (1.2) could be expressed in the form

$$I(h) = \lambda_1 J_1(h) + \lambda_2 J_2(h) + \cdots + \lambda_N J_N(h)$$

and where $J_1(h), J_2(h), \ldots, J_N(h)$ constituted a so-called *Tchebycheff system* on its domain of definition, a set of functions with properties similar to those of polynomials. Among many papers using this method to obtain bounds on the number of zeroes of some Abelian integrals we would stress the works of Petrov [**19, 20**] and Mardesic [**18**].

Before introducing an example of such use, let us remind its definition and list some of their most relevant properties.

2.1. Tchebycheff systems. Let us consider $\{g_0, g_2, \ldots, g_n\}$ a system of continuous functions defined on a Hausdorff space A. We call it a *Tchebycheff system*, T-system in short, if they satisfy the so-called *Haar condition*, i.e., for any $n+1$ distinct points $x_0,\ x_1, \ldots,\ x_n$ of A the vectors

$$\begin{pmatrix} g_0(x_0) \\ g_0(x_1) \\ \vdots \\ g_0(x_n) \end{pmatrix}, \quad \begin{pmatrix} g_1(x_0) \\ g_1(x_1) \\ \vdots \\ g_1(x_n) \end{pmatrix}, \quad \ldots \quad, \begin{pmatrix} g_n(x_0) \\ g_n(x_1) \\ \vdots \\ g_n(x_n) \end{pmatrix},$$

are linearly independent. In other words, the determinant

$$\begin{vmatrix} g_0(x_0) & g_1(x_0) & \cdots & g_n(x_0) \\ g_0(x_1) & g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ g_0(x_n) & g_1(x_n) & \cdots & g_n(x_n) \end{vmatrix} \neq 0 \tag{2.1}$$

for any $n+1$ distinct points $x_0, x_1, x_2, \ldots, x_n$ in A. The $n+1$-dimensional subspace of $\mathcal{C}(A)$ defined by $\{g_0, g_1, \ldots, g_n\}$, denoted by *span* $\{g_0, g_1, \ldots, g_n\}$, is called a *Tchebycheff space*.

This kind of spaces generalizes the behavior of the space of polynomials of degree less or equal than n, *span* $\{1, x, \ldots, x^n\}$ regarding the number of their zeroes and the uniqueness of solution of the interpolation problem. Precisely, we have (see [**1**, page 92]):

Proposition 2.1. *Let $g_0, g_1, \ldots, g_n$ be $n+1$ real (or complex) valued continuous functions defined on a Hausdorff space A containing, at least, $n+1$ points. Then the following assertions are equivalent:*

- (i) *Any non-zero f belonging to span $\{g_0, g_1, \ldots, g_n\}$, the subspace generated by g_0, $g_1, \ldots, g_n$, has at most n distinct zeros in A. Multiple zeroes must be considered with their multiplicity, that is, a double zero counts like two zeroes and so on.*
- (ii) *The* interpolation problem *has a unique solution: for any $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in A and real (or complex) numbers $y_0, y_1, \ldots, y_n$, there exists a unique $f \in$ span $\{g_0, g_1, \ldots, g_n\}$ satisfying that*

$$f(x_j) = y_j, \qquad j = 0, 1, \ldots, n.$$

- (iii) *If $x_0, x_1, \ldots, x_n$ are distinct points of A, then the determinant*

$$\begin{vmatrix} g_0(x_0) & \cdots & g_n(x_0) \\ \vdots & \ddots & \vdots \\ g_0(x_n) & \cdots & g_n(x_n) \end{vmatrix} \neq 0$$

Despite there are no many examples of T-systems, one can find interesting cases in the literature.

Lemma 2.2. *The following sets of functions are T-systems:*

1. $\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$ *for $x \in [0, \infty)$, provided that $0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$. It is also a T-system on $(0, \infty)$ if $\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$.*
2. $\{x^{\lambda_0}, x^{\lambda_0}\log x, x^{\lambda_1}, x^{\lambda_1}\log x, \ldots, x^{\lambda_n}, x^{\lambda_n}\log x\}$ *in $(0, \infty)$ provided $\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$.*
3. *If $\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$, the system*

$$\left\{\frac{1}{x-\lambda_0}, \frac{1}{x-\lambda_1}, \ldots, \frac{1}{x-\lambda_n}\right\},$$

for $x \in \mathbb{R} \setminus \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$.

4. $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\}$ *for $x \in [0, 2\pi)$ and* $\{1, \cos x, \cos 2x, \ldots, \cos nx\}$ *for $x \in [0, \pi)$.*

On the other hand, the following useful properties satisfied by T-systems are known:

LEMMA 2.3. *Let A be a Hausdorff space with at least $n+1$ distinct points. Then:*

- (i) *If $\{g_0, g_1, \ldots, g_n\}$ is a T-system on A then it is also a T-system on any subset $B \subset A$, provided B contains at least $n+1$ distinct points.*
- (ii) *If $\{g_0, \ldots, g_n\}$ is a T-system on A and $f \in \mathcal{C}(A)$ has constant sign on A, then $\{fg_0, fg_1, \ldots, fg_n\}$ is also a T-system on A.*
- (iii) *If $\{g_0, g_1, \ldots, g_n\}$ is a T-system on $[a,b]$ then*

$$\left\{1, \int_a^x g_0(t)\,dt, \int_a^x g_1(t)\,dt, \ldots, \int_a^x g_n(t)\,dt, \right\}$$

is also a T-system on $[a,b]$, provided all $g_j(x)$ are Lebesgue integrable on $[a,b]$.

- (iv) *The same result holds for*

$$\left\{1, c_0 + \int_a^x g_0(t)\,dt, c_1 + \int_a^x g_1(t)\,dt, \ldots, c_n + \int_a^x g_n(t)\,dt, \right\}$$

$c_0, c_1, \ldots, c_n$ *being arbitrary real constants.*

For more details on T-systems we refer the reader, for instance, to the books of Karlin and Studden [**13**], Cheney [**3**], Borwein and Erdélyi [**1**] and references therein.

2.2. An example. The problem of determining the number of limit cycles bifurcating from the period annulus of systems of type

$$\begin{cases} \dot{x} &=& -yF(x,y) + \varepsilon\, P(x,y) \\ \dot{y} &=& xF(x,y) + \varepsilon\, Q(x,y) \end{cases},$$

for ε small, $P(x,y)$, $Q(x,y)$ polynomials and F satisfying $F(0,0) \neq 0$, has been widely studied (see, for instance, [**15, 24, 25, 4, 16, 9, 2, 8**]). In this example we will consider systems of the form

$$\begin{cases} \dot{x} &=& -yF_K(x) + \varepsilon\, P_n(x,y) \\ \dot{y} &=& xF_K(x) + \varepsilon\, Q_n(x,y) \end{cases}, \tag{2.2}$$

$P_n(x,y)$, $Q_n(x,y)$ being polynomials of degree n and $F_K(x)$ consisting on a set of vertical straight lines $x = \mathrm{a}_j$, $j = 1, \ldots, K$, that is,

$$F_K(x) = \prod_{j=1}^{K} (x - \mathrm{a}_j), \tag{2.3}$$

where $\{\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_K\}$ are distinct positive real numbers. By construction, any line $x = \mathrm{a}_j$ becomes an invariant set of singular points.

A possible application of this method is to look for estimates on the number of limit cycles which can appear from bifurcations of periodic orbits of the unperturbed system ($\varepsilon = 0$) covering the period annulus

$$\mathcal{D} = \left\{ (x,y) \in \mathbb{R}^2 \mid 0 < \sqrt{x^2 + y^2} < \mathrm{a} = \min_j \mathrm{a}_j \right\}.$$

This type of problems has been studied in many papers presenting different choices for the invariant sets of singulars points (isolated points, straight lines with different multiplicities, conics, etc) [**24, 25, 2, 8**].

As it is standard in this kind of problems, we can rewrite our system (2.2) in $\mathcal{D}$ in the equivalent form

$$\begin{cases} \dot{x} &= -y + \varepsilon\, P_n(x,y) / \displaystyle\prod_{j=1}^{K} (x - \mathrm{a}_j) \\ \dot{y} &= x + \varepsilon\, Q_n(x,y) / \displaystyle\prod_{j=1}^{K} (x - \mathrm{a}_j) \end{cases} . \tag{2.4}$$

and to denote by

$$\gamma_r = \left\{ (x,y) \mid x^2 + y^2 = r^2 \right\}, \quad \text{for } 0 < r < \mathrm{a}_1$$

any periodic orbit in this period annulus of the unperturbed system. As it was mentioned in the introduction, isolated zeroes of the displacement map are intimately related to the number of limit cycles bifurcating from the family of ovals $\{\gamma_r\}$ and, in particular, to the maximum number of zeroes of the associated Abelian integral

$$I(r) = \int_{\gamma_r} \frac{Q(x,y)\, dx - P(x,y)\, dy}{F_K(x,y)}. \tag{2.5}$$

One of the aims in the paper [**7**] was to provide an upper bound for the number of zeroes of this Abelian Integral associated to a system (2.2), depending on the number K of critical straight lines and the degree n of the perturbation polynomials P_n and Q_n.

Thus, in [**7**] it is proved the following result:

THEOREM 2.4. *Let us consider a system of the form (2.2),*

$$\begin{cases} \dot{x} &= -yF_K(x) + \varepsilon\, P_n(x,y) \\ \dot{y} &= xF_K(x) + \varepsilon\, Q_n(x,y) \end{cases},$$

where

$$F_K(x) = \prod_{j=1}^{K} (x - \mathrm{a}_j),$$

with real positive numbers $\{\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_K\}$*, are vertical straight lines of singular points and* P_n*,* Q_n *are polynomials in* x, y *of degree* n*.* ε *is an small parameter. For such a system, let us consider the associated Abelian Integral*

$$I(r) = \int_{\gamma_r} \frac{Q_n(x,y)\, dx - P_n(x,y)\, dy}{\displaystyle\prod_{j=1}^{K} (x - \mathrm{a}_j)}.$$

Then, for $K \geq 1$*,* $n \geq 1$*, we have that the number of zeroes of* $I(h)$*, denoted by* $\mathcal{Z}(I)$*, can be bounded by*

$$\mathcal{Z}(I) \leq (K+1)\left(\left[\frac{n}{2}\right] + 1\right) + 1. \tag{2.6}$$

Therefore, since $\mathcal{Z}(I)$ is an upper bound of $\mathcal{Z}_S(I)$, the number of simple zeroes of I, one can deduce that, for $\varepsilon \neq 0$ small enough, the number of limit cycles bifurcating from periodic orbits of the unperturbed system is bounded by

$$\mathcal{Z}(I) \leq (K+1)\left(\left[\frac{n}{2}\right] + 1\right) + 1.$$

Have in mind, however, that this bound is not optimal.

The proof starts checking that our Abelian integral $I(r)$ admits an expression of the form

$$I(r) = \int_{\gamma_r} \frac{Q_n(x,y)\,dx - P_n(x,y)\,dy}{\prod_{j=1}^{K}(x-\mathrm{a}_j)} = \varphi^{(0)}_{[n/2]}(r^2) + \sum_{j=1}^{K} \frac{\varphi^{(j)}_{[(n+1)/2]}(r^2)}{\sqrt{\mathrm{a}_j^2 - r^2}},$$

where $\varphi_s^{(m)}(\rho)$ are polynomial of degree s in ρ and $[z]$ denotes the integer part of z. After that (with some effort) one can prove that these functions form a Tchebycheff vector space and applying results in subsection 2.1 the bound (2.6) is derived.

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