RESEARCH ARTICLE | JANUARY 24 2024

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J. Math. Phys. 65, 012106 (2024)
https://doi.org/10.1063/5.0169069

Journal of Mathematical Physics ARTICLE pubs.aip.org/aip/jmp

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Cite as: J. Math. Phys. 65, 012106 (2024); doi: 10.1063/5.0169069
Submitted: 22 July 2023 • Accepted: 13 December 2023 •
Published Online: 24 January 2024

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ABSTRACT

In this work we explain the relevance of the Differential Galois Theory in the semiclassical (or WKB) quantification of some two degree of freedom potentials. The key point is that the semiclassical path integral quantification around a particular solution depends on the variational equation around that solution: a very well-known object in dynamical systems and variational calculus. Then, as the variational equation is a linear ordinary differential system, it is possible to apply the Differential Galois Theory to study its solvability in closed form. We obtain closed form solutions for the semiclassical quantum fluctuations around constant velocity solutions for some systems like the classical Hermite/Verhulst, Bessel, Legendre, and Lamé potentials. We remark that some of the systems studied are not integrable, in the Liouville–Arnold sense.

I. INTRODUCTION

In Ref. 1 the third author suggested the relevance that the Differential Galois Theory could play in the Feynman's path integral approach in Quantum Mechanics. Indeed, the key proposal was to study whether it was possible to obtain, in closed form, the semiclassical approximation of the Feynman's propagator K, see Refs. 2 and 3.

Let us recall the notation and main ideas in Ref. 1. Given n the number of degrees of freedom, we denote by $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ the position, t the time, and γ is a path from $(\mathbf{x}_0, t_0)$ to $(\mathbf{x}_1, t_1)$. This classical path γ in the configuration space defines an integral curve Γ in the phase space, assuming there are non focal (conjugated) points (Chap. 9 of Ref. 4).

The computation of the propagator $K(\mathbf{x}_1, t_1|\mathbf{x}_0, t_0)$ around the path γ in the semiclassical approach (where $\hbar$ is small) can be obtained through

$$K(\mathbf{x}_1, t_1|\mathbf{x}_0, t_0) = K_{\mathrm{WKB}}(1 + O(\hbar)),$$

K_{WKB} being the semiclassical approximation of the propagator K (WKB after Wentzel, Kramers, and Brillouin, 1926). The function K_{WKB} is given by the so-called *Pauli–Morette formula*

$$K_{\mathsf{WKB}}(\mathbf{x}_1, t_1|\mathbf{x}_0, t_0) = A\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\gamma)}, \qquad \text{with} \qquad A = \frac{1}{(2\pi i\hbar)^{n/2}}\frac{1}{\sqrt{\det J(t_1, t_0)}}, \tag{1}$$

called prefactor, where:

- The *fixed classical path* γ parametrized by $(\mathbf{x}(t), t)$, for $t_0 \le t \le t_1$, with starting point $(\mathbf{x}_0, t_0)$ and endpoint $(\mathbf{x}_1, t_1)$, assuming no focal points.
- S is the *action* computed on this classical path γ, i.e.,

$$S[\mathbf{x}(t)] := S(\gamma) = \int_{t_0}^{t_1} L(\mathbf{x}, \dot{\mathbf{x}}, t)\ dt = \int_{t_0}^{t_1}\left(\sum_{i=1}^{n} y_i\dot{x}_i - H(\mathbf{x}, \mathbf{y}, t)\right) dt, \tag{2}$$

  with $L(\mathbf{x}, \dot{\mathbf{x}}, t)$ and $H(\mathbf{x}, \mathbf{y}, t)$ being the corresponding *Lagrangian* and *Hamiltonian* functions.
- The $n \times n$ matrix $J = J(t_1, t_0)$ is given by a block inside a fundamental matrix $\Psi(t, t_0)$ of the variational equations around the phase integral curve defined by the classical path γ:

$$\Psi(t, t_0) = \begin{pmatrix} \circ & J(t, t_0) \\ \circ & \circ \end{pmatrix}.$$

  That is, if the fundamental matrix (with initial condition $\Psi(t_0, t_0) = \mathrm{Id}_{2n}$, the $2n$-dimensional identity matrix) is splitted into four square boxes of dimension n, the matrix J is given by the variation of the positions with respect to the initial momenta. We will refer to J and $\det J$ as the *Van Vleck–Morette matrix and its determinant*, respectively.

We consider the above connection between the semiclassical propagator and the variational equation as a quantum mechanical confirmation of the following fundamental *Bryce DeWitt's Principle*:[11]

"The quantum theory is basically a theory of small disturbances"

So, the computation of the semiclassical propagator in formula (1) is based on the matrix J, obtained in its turn by means of the solution of the variational equation. Recall that the variational equation is also called Jacobi equation (in the context of variational calculus), equation of geodesic dispersion (in general relativity) or equation of small disturbances (in agreement with Bryce DeWitt's Principle). It is easy to see that

$$J(t_1, t_0) = \left(\frac{\partial\xi_i(t_1)}{\partial\eta_j(t_0)}\right), \tag{3}$$

where $(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$ are the variables in the corresponding variational equation, and $\xi_i = \delta x_i$, $\eta_i = \delta y_i$ being the variations in positions and momenta, respectively. Since the variational equation is a linear differential system it is possible to study its solutions by means of the differential Galois theory (see Ref. 12 for details).[5–10] In the case that the classical Hamiltonian system under consideration is integrable in *Liouville–Arnold* sense then one of the main results in Ref. 1 guarantees that it is possible to obtain a closed form formula for K_{WKB}, in a very precise way. Keep in mind, however, that this is only a necessary condition for integrability. Indeed, as it will be seen, some of the Hamiltonian families considered along this paper are not integrable *but admit closed analytic formulas for the semiclassical propagator* around some special classical paths.

Along this paper integrability of the variational equation means integrability in the sense of the differential Galois theory. This kind of integrability is characterized by the structure of the Galois group of the equation: the identity component of the Galois group must be solvable (see the Appendix of Ref. 1 and references therein).

One of the open problems posed in Ref. 1 was to compute this propagator for concrete families of Hamiltonian systems with several degrees of freedom. This paper aims to be a first work filling this gap.

To do that, we consider two-degrees of freedom Hamiltonian systems and compute the propagator along classical paths γ defined by invariant planes in the phase space, whose restricted dynamics fall in a free particle model of one-degree of freedom. That is, defined by

$$E = \frac{p_x^2}{2}, \quad p_x = \dot{x},$$

whose solution curve is given by $(x_E(t), p_{x_E}(t))$, where

$$x_E(t) = \sqrt{2E}t + x_0, \qquad p_{x_E}(t) = \sqrt{2E}.$$

Since only autonomous Hamiltonian systems are considered we can, without loss of generality, take initial time $t_0 = 0$. Therefore, for $n = 2$ the Pauli–Morette formula (1) reads

$$K_{\mathrm{WKB}}(\mathbf{x}_1, t_1 \mid \mathbf{x}_0, 0) = \frac{1}{2\pi i\hbar}\frac{1}{\sqrt{\det J(t_1, 0)}}\mathrm{e}^{\frac{i}{\hbar}S(\gamma)}. \tag{4}$$

We want to stress that we shall only semiclassically quantify around this very special type of solutions, i.e., those of a free particle motion, their most important quantum oscillations occurring in their transversal directions. We do not know the physical relevance of this kind of quantification around such a path. In any case, to quantify around particular special solutions is still today a very frequent method in quantum mechanics and in quantum field theory. For instance, the case around *instantons* in tunneling problems.

Furthermore, is very likely that for the families of potentials considered here it would be impossible to quantify by means of closed form formulas around an arbitrary classical path. The reasons are clear: before quantifying, it is necessary to get an analytic expression for the particular integral curve of the classical mechanical system (or the associate classical path). Indeed, as mentioned above, some of the families considered here are not integrable, and hence, such general solution in closed form does not exist. Thus, we state the following results (for further details see Sec. IV):

Consider the families of potentials defined by Eq. (5) with $k = 2$ given by Table (17). Then, the Bessel and Legendre families with $b \neq 0$ of Hamiltonian systems, as well as Hermite and Lamé families are not integrable in the Liouville–Arnold sense. Furthermore the semiclassical approximations of the corresponding Feynman propagators are not integrable.

We remark that closed form solutions, also in the framework of the differential Galois theory, of some non-autonomous one-dimensional oscillators were obtained by the first named author in Refs. 13–15.

The paper is structured as follows: Sec. II is devoted to some generalities regarding the structure of the Van Vleck–Morette determinant for two-degrees of freedom Hamiltonian systems around invariant planes with free particle reduced dynamics. In Sec. III we obtain closed analytical formulas for the Van Vleck–Morette determinant and hence the semiclassical approximation of the Feynman propagators for four classical families of potentials: Hermite, Bessel, Legendre and Lamé, see Table (17). For any family, an illustrative example accompanying the theoretical result is provided. Finally, in Sec. IV we prove Proposition 1 and Corollary 1.

II. POTENTIALS WITH FREE PARTICLE MOTION IN AN INVARIANT PLANE

Let us consider classical Hamiltonian systems with $n = 2$ degrees of freedom. Assume that they are given by the sum of a kinetic energy T and a certain kind of potentials V. More precisely, if (x, y) stands for the spatial variables and (p_x, p_y) for the corresponding momenta, then the Hamiltonian function is $H = H(x, p_x, y, p_y) = T(p_x, p_y) + V(x, y)$, where

$$T(p_x, p_y) = \frac{p_x^2 + p_y^2}{2},$$

and

$$V(x, y) = y^k f(x, y), \quad f(x, 0) \neq 0,\ k \in \mathbb{N},\ k \geq 2. \tag{5}$$

Despite of its concrete form, this kind of potentials has been considered in references as Refs. 12 and 16 among others. The system of ordinary differential equations associated to $H(x, p_x, y, p_y)$ reads

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = p_x, & \dot{y} &= \frac{\partial H}{\partial p_y} = p_y, \\ \dot{p}_x &= -\frac{\partial H}{\partial x} = -y^k\frac{\partial f}{\partial x}(x, y), & \dot{p}_y &= -\frac{\partial H}{\partial y} = -ky^{k-1}f(x, y) - y^k\frac{\partial f}{\partial y}(x, y), \end{aligned} \tag{6}$$

where dot means derivative with respect to t. This kind of Hamiltonian systems has, for $k \geq 2$, the invariant plane $\Gamma = \{y = p_y = 0\}$ (in the phase space), with associated $\{y = 0\}$ invariant straight line path γ in the configuration space.

We restrict ourselves to solutions of (6) lying on Γ, which connect two points $(x_0, p_{x_0}, 0, 0)$ and $(x_1, p_{x_1}, 0, 0)$, at times $t_0 = 0$ and $t = t_1$, respectively, and such that its motion is the one of a free particle, that is,

$$x(t) = \frac{x_1 - x_0}{t_1}\, t + x_0, \qquad p_x(t) = \frac{x_1 - x_0}{t_1}. \tag{7}$$

Fixed an energy level $H(x, p_x, y, p_y) = E$, we denote by γ_E the free particle path (7) associated to this energy E. This implies, in particular, the relation $E = \frac{1}{2}p_x^2$. Hence,

$$E = H(x, \dot{x}, 0, 0) = \frac{1}{2}\dot{x}^2 \Longrightarrow \dot{x}(t) = \sqrt{2E} \Longrightarrow x(t) = \sqrt{2E}\ t + x(0),$$

it follows that the (free-particle) path γ_E can be parameterized by

$$x_E(t) := \sqrt{2E}\,t + x_0, \quad x_0 = x(0). \tag{8}$$

Moreover,

$$E = \frac{1}{2}\left(\frac{x_1 - x_0}{t_1}\right)^2 > 0,$$

from which the action on γ_E becomes

$$\begin{aligned} S[\gamma_E] &= \int_{t_0=0}^{t_1} (p_x\dot{x} + p_y\dot{y} - H(x,p_x,y,p_y))\,dt = \int_0^{t_1}\left(p_x^2(t) - \frac{1}{2}p_x^2(t)\right)dt \\ &= \int_0^{t_1} E\,dt = Et_1 = \frac{1}{2}\frac{(x_1 - x_0)^2}{t_1}. \end{aligned}$$

To avoid misunderstandings, henceforward we will fix the following order (x, p_x, y, p_y) for the variables, and will denote by $X_H = (F_1, F_2, F_3, F_4)$ the right hand-side vector field in (6). In general, if Γ is any solution of (6) then the corresponding *variational equation* around Γ is defined as

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\eta}_1 \\ \dot{\xi}_2 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial p_x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial p_y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial p_x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial p_y} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial p_x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial p_y} \\ \frac{\partial F_4}{\partial x} & \frac{\partial F_4}{\partial p_x} & \frac{\partial F_4}{\partial y} & \frac{\partial F_4}{\partial p_y} \end{pmatrix}\Bigg|_{\Gamma} \begin{pmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \end{pmatrix},$$

where ξ_j stands for the positions and η_j for the momenta. In the case of system (6), the variational equation around any solution lying on Γ takes the form

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\eta}_1 \\ \dot{\xi}_2 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \mathcal{A}_1(x,y) & 0 & \mathcal{A}_2(x,y) & 0 \\ 0 & 0 & 0 & 1 \\ \mathcal{A}_2(x,y) & 0 & \mathcal{A}_3(x,y) & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \end{pmatrix}, \tag{9}$$

with

$$\begin{aligned} \mathcal{A}_1(x,y) &= -y^k\frac{\partial^2 f}{\partial x^2}(x,y), \qquad \mathcal{A}_2(x,y) = -ky^{k-1}\frac{\partial f}{\partial x}(x,y) - y^k\frac{\partial^2 f}{\partial x \partial y}(x,y), \\ \mathcal{A}_3(x,y) &= -k(k-1)y^{k-2}f(x,y) - 2ky^{k-1}\frac{\partial f}{\partial y}(x,y) - y^k\frac{\partial^2 f}{\partial y^2}(x,y), \end{aligned}$$

for $k \geq 2$.

On one hand, the case $k > 2$ becomes

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\eta}_1 \\ \dot{\xi}_2 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \end{pmatrix},$$

with straightforward solutions.

On the other, the case $k = 2$ is much richer and is, therefore, the one tackled in this work.

Indeed, for $k = 2$ variational equation reduces to

$$\begin{pmatrix}\dot{\xi}_1\\ \dot{\eta}_1\\ \dot{\xi}_2\\ \dot{\eta}_2\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -2f(x_E(t),0) & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \eta_1\\ \xi_2\\ \eta_2\end{pmatrix}. \tag{10}$$

This system can be divided in two components: the *tangential* one, with variables (ξ_1, η_1); and the *normal* one, normal variational equation, with variables (ξ_2, η_2). Notice that these two systems of variational equations appear uncoupled, fact which simplifies its resolution. Certainly, the tangential variational equation is $\dot{\xi}_1 = \eta_1,\ \dot{\eta}_1 = 0$, whose solution is

$$\xi_1(t) = \eta_1(0)t + \xi_1(0), \quad \eta_1(t) = \eta_1(0).$$

On the other hand, the normal variational equation always reduces to

$$\dot{\xi}_2 = \eta_2, \qquad \dot{\eta}_2 = -2f(x_E(t),0)\ \xi_2,$$

or, equivalently,

$$\ddot{\xi}_2 + 2f(x_E(t),0)\xi_2 = 0, \qquad \eta_2 = \dot{\xi}_2. \tag{11}$$

If we denote by

$$\Phi(t) = \begin{pmatrix}\phi_{11}(t) & \phi_{12}(t)\\ \phi_{21}(t) & \phi_{22}(t)\end{pmatrix}, \tag{12}$$

the fundamental matrix solution of the normal variational equation satisfying that $\Phi(0) = \mathrm{Id}_2$, the identity matrix, then we have that

$$\begin{pmatrix}\xi_2(t)\\ \eta_2(t)\end{pmatrix} = \begin{pmatrix}\phi_{11}(t) & \phi_{12}(t)\\ \phi_{21}(t) & \phi_{22}(t)\end{pmatrix}\begin{pmatrix}\xi_2(0)\\ \eta_2(0)\end{pmatrix}.$$

Therefore,

$$\frac{\partial \xi_1(t)}{\partial \eta_1(0)} = t, \qquad \frac{\partial \xi_1(t)}{\partial \eta_2(0)} = 0, \qquad \frac{\partial \xi_2(t)}{\partial \eta_1(0)} = 0, \qquad \frac{\partial \xi_2(t)}{\partial \eta_2(0)} = \phi_{12}(t),$$

and

$$\xi_2(t) = c_1\xi_2^{(1)}(t) + c_2\xi_2^{(2)}(t),$$

where $\{\xi_2^{(1)}, \xi_2^{(2)}\}$ is a basis of solutions of (11). So, from the formula (3), the Van Vleck–Morette matrix $J(t_1, t_0 = 0)$ reads

$$J(t_1,0) = \begin{pmatrix}\dfrac{\partial \xi_1(t_1)}{\partial \eta_1(0)} & \dfrac{\partial \xi_1(t_1)}{\partial \eta_2(0)}\\ \dfrac{\partial \xi_2(t_1)}{\partial \eta_1(0)} & \dfrac{\partial \xi_2(t_1)}{\partial \eta_2(0)}\end{pmatrix} = \begin{pmatrix}t_1 & 0\\ 0 & \phi_{12}(t_1)\end{pmatrix}, \tag{13}$$

and has determinant $\det J(t_1,0) = t_1\phi_{12}(t_1)$. From now on, we refer as variational equation the normal variational equation. Thus, the semiclassical approximation of the propagator along the solution γ_E, given by the Pauli–Morette formula (1), is

$$K_{\mathrm{WKB}}(x_1, t_1 \mid x_0, 0) = \frac{1}{2\pi i\hbar}\ \frac{1}{\sqrt{t_1\phi_{12}(t_1)}}\,\mathrm{e}^{\frac{i}{2\hbar t_1}(x_1-x_0)^2}, \tag{14}$$

As the path is on the line $y = 0$ we will write, by abusing of notation, $K_{\mathrm{WKB}}(x_1, t_1 \mid x_0, 0)$ to indicate $K_{\mathrm{WKB}}(\mathbf{x}_1, t_1|\mathbf{x}_0, 0)$.

If $k > 2$, straightforward computations lead to $\phi_{12}(t_1) = t_1$ and therefore $\det J(t_1,0) = t_1^2$. Hence,

$$K_{\mathrm{WKB}}(x_1, t_1|x, 0) = \frac{1}{2\pi i\hbar t_1}\ \mathrm{e}^{\frac{i}{2\hbar t_1}(x_1-x_0)^2},$$

which is the well known expression for the free particle propagator in two degrees of freedom.

In the case $k = 2$, the function $\phi_{12}(t_1)$ in (13) can be determined from the values at $t_0 = 0$ and t_1 of a basis of solutions, $\{\xi_2^{(1)}, \xi_2^{(2)}\}$. Precisely, in the general form

$$\xi_2(t) = c_1 \xi_2^{(1)}(t) + c_2 \xi_2^{(2)}(t), \qquad \eta_2(t) = \dot{\xi}_2(t) = c_1 \dot{\xi}_2^{(1)}(t) + c_2 \dot{\xi}_2^{(2)}(t),$$

the values c_1 and c_2 are uniquely determined from the initial conditions at $t = 0$ and so they are functions of $\xi_2(0)$ and $\eta_2(0)$ or, equivalently, of $\xi_2^{(j)}(0)$ and $\dot{\xi}_2^{(j)}(0)$, for $j = 1, 2$. So,

$$\phi_{12}(t) = \frac{\partial \xi_2(t)}{\partial \eta_2(0)} = \frac{\partial \xi_2(t)}{\partial c_1} \cdot \frac{\partial c_1}{\partial \eta_2(0)} + \frac{\partial \xi_2(t)}{\partial c_2} \cdot \frac{\partial c_2}{\partial \eta_2(0)} = \xi_2^{(1)}(t) \cdot \frac{\partial c_1}{\partial \eta_2(0)} + \xi_2^{(2)}(t) \cdot \frac{\partial c_2}{\partial \eta_2(0)}. \tag{15}$$

To compute these two partial derivatives, we solve the linear system

$$\xi_2(0) = c_1 \xi_2^{(1)}(0) + c_2 \xi_2^{(2)}(0), \qquad \eta_2(0) = c_1 \dot{\xi}_2^{(1)}(0) + c_2 \dot{\xi}_2^{(2)}(0),$$

by Cramer's rule (because it has a unique solution) and get

$$c_1 = \frac{1}{D}\Big(\xi_2(0)\dot{\xi}_2^{(2)}(0) - \eta_2(0)\xi_2^{(2)}(0)\Big), \qquad c_2 = \frac{1}{D}\Big(\eta_2(0)\xi_2^{(1)}(0) - \xi_2(0)\dot{\xi}_2^{(1)}(0)\Big),$$

where

$$D = \begin{vmatrix} \xi_2^{(1)}(0) & \xi_2^{(2)}(0) \\ \dot{\xi}_2^{(1)}(0) & \dot{\xi}_2^{(2)}(0) \end{vmatrix}.$$

Consequently,

$$\frac{\partial c_1}{\partial \eta_2(0)} = -\frac{\xi_2^{(2)}(0)}{D}, \qquad \frac{\partial c_2}{\partial \eta_2(0)} = \frac{\xi_2^{(1)}(0)}{D},$$

and substituting into expression (15) we obtain

$$\phi_{12}(t_1) = \frac{\partial \xi_2(t_1)}{\partial \eta_2(0)} = \frac{1}{D}\begin{vmatrix} \xi_2^{(1)}(0) & \xi_2^{(1)}(t_1) \\ \xi_2^{(2)}(0) & \xi_2^{(2)}(t_1) \end{vmatrix}.$$

Since the relevant variational equation in our study is the normal one (already denoted by, just, variational), from now we will remove from its basis of solutions the subscript 2, that is, $\xi_2^{(j)}$ will be referred, simply, as $\xi^{(j)}$, $j = 1, 2$. Thus, summarising, for $k = 2$ our semiclassical approximate propagator K_{WKB} is given by the formula (14), where

$$\phi_{12}(t_1) = \frac{1}{D}\begin{vmatrix} \xi^{(1)}(0) & \xi^{(1)}(t_1) \\ \xi^{(2)}(0) & \xi^{(2)}(t_1) \end{vmatrix}, \quad D = \begin{vmatrix} \xi^{(1)}(0) & \xi^{(2)}(0) \\ \dot{\xi}^{(1)}(0) & \dot{\xi}^{(2)}(0) \end{vmatrix}, \tag{16}$$

and $\{\xi^{(1)}, \xi^{(2)}\}$ being a fundamental solution of the variational Eq. (11)

$$\ddot{\xi} + 2f(x_E(t), 0)\ \xi = 0,$$

with $x_E(t) = \sqrt{2E}\ t + x_0$. Dot denotes derivative with respect to t.

This notation will be maintained henceforth in the paper.

III. APPLICATIONS

This section is devoted to the application of this result to some relevant families of equations. The type of the function $2f(x, 0)$ appearing in the variational Eq. (11) determines the family, according to the following table:

$2f(x,0)$	Family to which it reduces
$1 - ax^2$	Hermite
$b - \dfrac{a}{x^2}$	Bessel
$-b + \dfrac{a}{\cosh^2 x}$	Legendre
$-b - a\wp(x + \omega_3)$	Lamé

(17)

where in all four cases a, b are real parameters. Here $\wp$ is the Weierstrass function with real period $2\omega_1$ and imaginary period $2\omega_3$.

In the case $a = 0$ all the potentials given in the table above have variational equations which reduce to a constant coefficients ode:

$$\ddot{\xi} + \omega\xi = 0, \quad \omega \in \mathbb{R}.$$

In particular the K_{WKB} is integrable (in the sense of the Differential Galois theory) for any value of the energy E. For them, we have:

ω	$\{\xi^{(1)}, \xi^{(2)}\}$	$\phi_{12}(t_1)$	$\det J$	$K_{\mathrm{WKB}}(x_1, t_1 \mid x_0, 0)$
0	$\{1, t\}$	t_1	t_1^2	$\frac{1}{2\pi i\hbar}\frac{1}{\lvert t_1\rvert}e^{\frac{i}{2\hbar t_1}(x_1-x_0)^2}$
> 0	$\{\cos\sqrt{\omega}\,t, \sin\sqrt{\omega}\,t\}$	$\frac{\sin\sqrt{\omega}\,t_1}{\sqrt{\omega}}$	$\frac{t_1\sin\sqrt{\omega}\,t_1}{\sqrt{\omega}}$	$\frac{1}{2\pi i\hbar}\frac{\omega^{1/4}}{\sqrt{t_1\sin\sqrt{\omega}\,t_1}}e^{\frac{i}{2\hbar t_1}(x_1-x_0)^2}$
< 0	$\{\cosh\sqrt{-\omega}\,t, \sinh\sqrt{-\omega}\,t\}$	$\frac{\sinh\sqrt{-\omega}\,t_1}{\sqrt{-\omega}}$	$\frac{t_1\sinh\sqrt{-\omega}\,t_1}{\sqrt{-\omega}}$	$\frac{1}{2\pi i\hbar}\frac{(-\omega)^{1/4}}{\sqrt{t_1\sinh\sqrt{-\omega}\,t_1}}e^{\frac{i}{2\hbar t_1}(x_1-x_0)^2}$

Considering $V(x, y) = \frac{\omega}{2}y^2$, the case $\omega = 0$ is the free particle case and $\omega > 0$ is the harmonic oscillator, which already is considered in the seminal works of Feynman, see for instance, Refs. 17 and 18. Thus, the semiclassical approximation is the complete approximation: $K_{\mathrm{WKB}} = K$.

Henceforth in the paper, we will assume $a \neq 0$.

A. Variational equations with Hermite equation. Verhulst potentials

Potentials of the form

$$V(x, y) = y^2 f(x, y), \quad f(x, y) = \frac{1}{2}(1 - ax^2) + \text{h.o.t.}(y),$$

where h.o.t.(y) means higher order terms in y. These systems are common in many physical systems. One example is the so-called Verhulst's potentials, which take the form

$$V(x, y) = \frac{1}{2}\left(\omega_1^2 x^2 + \omega_2^2 y^2\right) - \left(\frac{A_1}{3}x^3 + A_2 xy^2\right) - \left(\frac{B_1}{4}x^4 + \frac{B_2}{2}x^2y^2 + \frac{B_3}{4}y^4\right), \tag{18}$$

with $\omega_1, \omega_2, A_1, A_2, B_1, B_2$, and B_3, real parameters. They were introduced by Verhulst[19] to study systems of axi-symmetric galaxies. They exhibit a discrete-symmetric potential and can undergo resonances of type 1:2, 1:1, 2:1 and 1:3. A suitable choice of the parameters and some trivial algebraic manipulations can lead them to fall into our class of potentials. For instance, swapping the variables x and y,

$$\tilde{V}(x, y) = V(y, x) = \frac{1}{2}\left(\omega_1^2 y^2 + \omega_2^2 x^2\right) - \left(\frac{A_1}{3}y^3 + A_2 yx^2\right) - \left(\frac{B_1}{4}y^4 + \frac{B_2}{2}y^2x^2 + \frac{B_3}{4}x^4\right),$$

and taking $\omega_1 = 1$, $\omega_2 = 0$, $A_1 = a_1$, $A_2 = 0$, $B_1 = b_1$, $B_2 = a$, and $B_3 = 0$, one gets $\tilde{V}(x, y) = y^2 f(x, y)$, with

$$f(x, y) = \frac{1}{2} - \frac{a_1}{3}y - \frac{b_1}{4}y^2 - \frac{a}{2}x^2 \quad \text{and} \quad f(x, 0) = \frac{1}{2} - \frac{a}{2}x^2. \tag{19}$$

Notice that $\Gamma = \{y = p_y = 0\}$ is an invariant plane of this system and hence the assumptions of Sec. II are satisfied. The corresponding variational equation around Γ is given by

$$\frac{d^2\xi}{dt^2} = -2f(x_E(t), 0)\xi, \qquad \eta(t) = \frac{d\xi}{dt}(t), \tag{20}$$

with

$$\begin{aligned} -2f(x_E(t), 0) &= -1 + ax_E^2(t) = -1 + a\left(\sqrt{2E}\ t + x_0\right)^2 \\ &= 2aE\ t^2 \pm 2\sqrt{2E}\ ax_0\ t + \left(ax_0^2 - 1\right). \end{aligned} \tag{21}$$

The time transformation

$$s = \sqrt[4]{2Ea}\left(t \pm \frac{x_0}{\sqrt{2E}}\right) \tag{22}$$

brings Eq. (20) with (21) into the harmonic oscillator ode

$$\frac{d^2\xi}{ds^2}(s) = \left(s^2 - \lambda\right)\xi(s), \qquad \eta(s) = \sqrt[4]{2Ea}\,\frac{d\xi}{ds}(s), \tag{23}$$

where $\lambda = \frac{1}{\sqrt{2Ea}}$. Observe that this transformation is equivalent to say

$$s = \pm\sqrt[4]{\frac{a}{2E}}\,x_E(t),$$

and so s is proportional to the spatial position of $x_E(t)$. This change simplifies the form of the variational equation. From the differential Galois Theory, it is know that ode (23) admits Liouvillian solutions if and only if $\lambda = 2m + 1$, where $m \in \mathbb{N} \cup \{0\}$. This implies that the set of admissible ("Liouvillian," say) energies E is discrete and it is given by

$$E_m = \frac{1}{2a}\left(\frac{1}{2m+1}\right)^2, \quad m \in \mathbb{N} \cup \{0\}. \tag{24}$$

Moreover, from relation (1) it follows that $(x_1 - x_0)^2 = 2E_m t_1^2$ and so, fixed the initial position x_0, the unique admissible positions x_1 are those satisfying that

$$|x_1 - x_0| = \frac{t_1}{\sqrt{a}}\frac{1}{2m+1}, \quad m \in \mathbb{N} \cup \{0\}. \tag{25}$$

Regarding the solvability of (23), Galois Theory ensures their Liouvillian solutions to be of the form

$$\xi(s) = P_m(s)e^{-s^2/2}, \tag{26}$$

where $P_m(s)$ is a polynomial of degree m (which we can assume, without loss of generality, to be monic). These polynomials P_m satisfy the celebrated Hermite differential equation

$$P_m''(s) - 2sP_m'(s) + 2mP_m(s) = 0. \tag{27}$$

Its solutions P_m are called Hermite polynomials and, among other nice properties, they have the same parity as m. i.e., if m is even then $P_m(s)$ is an even function and if m is odd then $P_m(s)$ is an odd function (see Ref. 20).

From D'Alembert formula, we know that

$$\xi^{(2)}(s) = \xi^{(1)}(s)\int_0^s \frac{e^{z^2}}{P_m^2(z)}\,dz,$$

is another solution of (23) independent of $\xi^{(1)}$. Together, $\{\xi^{(1)}(s), \xi^{(2)}(s)\}$ form a fundamental solution. In order to simplify the computations, we will assume henceforth in this section that $x_0 = 0$ and hence

$$s = \sqrt[4]{2E_m a}\,t. \tag{28}$$

The case $x_0 \neq 0$ follows analogously.

In order to obtain an expression for $\phi_{12}(t_1)$ in $\det J(0, t_1)$ we apply formula (15). In this context it reads as follows:

$$\phi_{12}(t_1) = \frac{1}{D}\begin{vmatrix} \xi^{(1)}(0) & \xi^{(1)}(s_1) \\ \xi^{(2)}(0) & \xi^{(2)}(s_1) \end{vmatrix}, \tag{29}$$

where

$$D = \begin{vmatrix} \xi^{(1)}(t=0) & \xi^{(2)}(t=0) \\ \dot{\xi}^{(1)}(t=0) & \dot{\xi}^{(2)}(t=0) \end{vmatrix} = \frac{1}{\sqrt{2m+1}}\begin{vmatrix} \xi^{(1)}(s=0) & \xi^{(2)}(s=0) \\ \frac{d\xi^{(1)}}{ds}(s=0) & \frac{d\xi^{(2)}}{ds}(s=0) \end{vmatrix}, \tag{30}$$

and we have taken into account the relation between s and t given by (28) and that

$$\sqrt[4]{2E_m a} = \frac{1}{\sqrt{2m+1}}, \qquad s = \frac{t}{\sqrt{2m+1}}. \tag{31}$$

The parity of $P_m(s)$ determines (as it will be seen later), a separate study for m even and m odd.

1. Case m even

Recall that in this case $P_m(s)$ is always an even function. Thus, on one side,

$$\xi^{(1)}(s=0) = P_m(0) \neq 0, \qquad \xi^{(2)}(s=0) = 0,$$

and on the other, since $P_m'(s)$ is odd,

$$\frac{d\xi^{(1)}}{ds}(0) = 0, \qquad \frac{d\xi^{(2)}}{ds}(0) = \frac{1}{P_m(0)}.$$

substituting in (30) we get $D = \frac{1}{\sqrt{2m+1}}$. Moreover,

$$\xi^{(1)}(s_1) = P_m(s_1)\ \mathrm{e}^{-s_1^2/2}, \qquad \xi^{(2)}(s_1) = P_m(s_1)\ \mathrm{e}^{-s_1^2/2}\int_0^{s_1} \frac{\mathrm{e}^{z^2}}{P_m^2(z)}\ dz,$$

so in Eq. (29) it gives rise to

$$\phi_{12}(t_1) = \sqrt{2m+1}\ P_m(0)P_m(s_1)\mathrm{e}^{-\frac{t_1^2}{2(2m+1)}}\int_0^{\frac{t_1}{\sqrt{2m+1}}} \frac{\mathrm{e}^{z^2}}{P_m^2(z)}\ dz.$$

Consequently, the determinant of Van Vleck–Morette reads

$$\det J(0,t_1) = t_1\phi_{12}(t_1) = \sqrt{2m+1}P_m(0)P_m\left(\frac{t_1}{\sqrt{2m+1}}\right)\mathrm{e}^{-\frac{t_1^2}{2(2m+1)}}\int_0^{\frac{t_1}{\sqrt{2m+1}}} \frac{\mathrm{e}^{z^2}}{P_m^2(z)}\ dz. \tag{32}$$

2. Case m odd

Same as in the previous case, we consider $\xi^{(1)}(s) = P_m(s)\mathrm{e}^{-s^2/2}$ as one solution of (23). The second one, provided by D'Alembert formula, is taken as

$$\xi^{(2)}(s) = \xi^{(1)}(s)\int_{s_0}^{s} \frac{\mathrm{e}^{z^2}}{P_m^2(z)}\ dz, \tag{33}$$

with $s_0 \neq 0$ to avoid the singularity inside the integral at $s = 0$ (remind that since P_m is odd then $P_m(0) = 0$. Let us assume write $P_m(s) = s^m + \cdots + a_3 s^3 + a_1 s$, where $a_1 \neq 0$ for any odd m. Then, having in mind the Taylor expansion of the exponential function, we have that

$$\xi^{(1)}(s) = a_1 s\big(1 - \mathcal{O}(s^2)\big). \tag{34}$$

Moreover,

$$\frac{\mathrm{e}^{z^2}}{P_m^2(z)} = \frac{1}{a_1^2 z^2}\big(1 + (1-2\tilde{a}_3)z^2 + \mathcal{O}(z^4)\big),$$

where $\tilde{a}_3 = \frac{a_3}{a_1}$, and so

$$\int_{s_0}^{s} \frac{\mathrm{e}^{z^2}}{P_m^2(z)}\ dz = \left(-\frac{1}{a_1^2 s} + \frac{1-2\tilde{a}_3}{a_1^2}s + \mathcal{O}(s^3)\right) - C_0,$$

with

$$C_0 = -\frac{1}{a_1^2 s_0} + \frac{1-2\tilde{a}_3}{a_1^2}s_0 + \cdots,$$

is a constant obtained by evaluating any primitive function of $\frac{\mathrm{e}^{z^2}}{P_m^2(z)}$ at the point s_0. Having in mind (34) and substituting the latter expression in (33) it follows that

$$\begin{aligned}\xi^{(2)}(s) &= \big(a_1 s + a_3 s^3 + \mathcal{O}(s^5)\big)\cdot\left(1 - \frac{s^2}{2} + \mathcal{O}(s^4)\right)\cdot\left(\left(-\frac{1}{a_1^2 s} + \frac{1-2\tilde{a}_3}{a_1^2}s + \mathcal{O}(s^3)\right) - C_0\right)\\ &= -\frac{1}{a_1} - a_1 C_0 s + \left(-\frac{a_3}{a_1^2} + \frac{3}{2a_1} - \frac{2\tilde{a}_3}{a_1}\right)s^2 + \mathcal{O}(s^3).\end{aligned}$$

Since $a_1 \neq 0$, the limit at $s = 0$ is well defined. Indeed, $\lim_{s\to 0} \xi^{(2)}(s) = -\frac{1}{a_1}$. This implies that $\xi^{(2)}(s)$ admits an analytic extension in $\mathbb{C}$. Abusing of notation, we denote this extension with the same name and so we write

$$\xi^{(2)}(0) = -\frac{1}{a_1}.$$

We have now the ingredients to compute $\phi_{12}(t_1)$ through formulas (29) and (30). Indeed, evaluating at $s = 0$,

$$\xi^{(1)}(0) = P_m(0) = 0, \qquad \xi^{(2)}(0) = -\frac{1}{a_1}, \qquad \frac{d\xi^{(1)}}{ds}(0) = P'_m(0) = a_1, \qquad \frac{d\xi^{(2)}}{ds}(0) = -a_1 C_0,$$

it follows that $D = \frac{1}{\sqrt{2m+1}}$. Therefore,

$$\phi_{12}(t_1) = \sqrt{2m+1}\begin{vmatrix} 0 & \xi^{(1)}(s_1) \\ -\frac{1}{a_1} & \xi^{(2)}(s_1) \end{vmatrix} = \frac{\sqrt{2m+1}}{P'_m(0)}\, P_m\left(\frac{t_1}{\sqrt{2m+1}}\right) \mathrm{e}^{-\frac{t_1^2}{2(2m+1)}},$$

and so

$$\det J(0,t_1) = \frac{\sqrt{2m+1}}{P'_m(0)}\, P_m\left(\frac{t_1}{\sqrt{2m+1}}\right) t_1 \mathrm{e}^{-\frac{t_1^2}{2(2m+1)}}. \tag{35}$$

To illustrate this case we propose the following simple example.

Example. Let us assume our potential to be $V(x,y) = y^2 f(x,y)$ with

$$f(x,y) = \frac{1}{2} - \frac{b_1}{4}y^2 - \frac{a}{2}x^2.$$

The Hamiltonian function is

$$H(x,p_x,y,p_y) = \frac{p_x^2 + p_y^2}{2} + y^2 f(x,y),$$

and its associated (Verhulst) Hamiltonian system

$$\dot{x} = p_x, \qquad \dot{y} = p_y,$$
$$\dot{p}_x = xy^2, \qquad \dot{p}_y = -y\left(1 - b_1 y^2 - a x^2\right).$$

Let us fix an energy level $H = E > 0$ and consider a free particle moving on the invariant plane $\Gamma = \{y = p_y = 0\}$ between two different points $(x_0, p_{x_0}, 0, 0)$ (at time $t_0 = 0$) and $(x_1, p_{x_1}, 0, 0)$ (at time $t = t_1$). Its trajectory is given by $(x_E(t), p_E(t), 0, 0)$ where $x_E(t) = \sqrt{2E}\, t$ and $p_E(t) = \dot{x}_E(t) = \sqrt{2E}$. Therefore, there exists a discrete set of values of the energy

$$\left\{E = E_m = \frac{1}{2}\left(\frac{1}{2m+1}\right)^2 \mid m \in \mathbb{N} \cup \{0\}\right\},$$

for which the determinant of Van Vleck–Morette admits a closed expression in terms of Liouvillian functions. This determinant is given by (32) if m even and by (35) if m odd. In the particular case that $m = 0$, that is $E_0 = 1/2$, after simplification due to $t_1 = |x_1 - x_0|$, we obtain

$$K_{\mathrm{WKB}}(x_1, t_1 | x_0, 0) = \frac{1}{2\pi^{5/4}\mathrm{i}\hbar}\, \frac{\mathrm{e}^{t_1^2/4}}{\sqrt{t_1 \operatorname{erfi}(t_1)}}\, \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}|x_1 - x_0|},$$

where

$$\operatorname{erfi}(t) = -\mathrm{i}\operatorname{erf}(\mathrm{i}\, t_1) = \frac{2}{\sqrt{\pi}}\int_0^t \mathrm{e}^{s^2}\, ds,$$

is the imaginary error function.

B. Variational equations with Bessel functions

Rational potentials have been widely studied by many authors (see, for example, Ref. 16 and references therein). They can fall in the hypothesis of this work by considering, for instance, potentials V of the form

$$V(x,y) = y^2 f(x,y), \quad f(x,y) = \frac{b}{2} - \frac{a}{2x^2} + \text{h.o.t.}(y), \tag{36}$$

a, b being real parameters non-simultaneously vanishing. This kind of potentials have singularities, which in our case, correspond to $x = 0$. To avoid it, in our study we will assume that $0 \notin [x_0, x_1]$, the x-interval where our free particle moves.

By the using of the change of variable $\tau = x_E(t) = \sqrt{2E}t + x_0$ in the variational Eq. (11) we obtain

$$\frac{d^2\xi_2}{d\tau^2} = \frac{1}{2E} r(\tau)\xi_2, \quad r(\tau) = -2f(\tau, 0). \tag{37}$$

Following the procedure given in Sec. II, for a given value $E > 0$ of the energy and considering the free particle motion given by (8), we reduce the study to its variational Eq. (37), which now reads

$$\frac{\partial^2\xi}{\partial\tau^2} = \frac{1}{2E}\left(\frac{a}{\tau^2} - b\right)\xi. \tag{38}$$

Equation (38), for $b \neq 0$, is one of the equivalent expressions of the well-known Bessel differential equation. For further details about Bessel odes see, for instance, Subsection 1 of the Appendix and the classical Ref. 21.

Comparing (38) with the standard Bessel equation in normal form

$$\frac{\partial^2\xi}{\partial\tau^2} = \left(\frac{\nu(\nu+1)}{\tau^2} - \mu^2\right)\xi, \qquad \nu \in \mathbb{C}, \qquad \mu \neq 0, \tag{39}$$

we get

$$a = 2\nu(\nu+1)E, \qquad b = 2E\mu^2.$$

It is well known that Bessel Eq. (39) is integrable if and only if $\nu := n$ is an integer number (see Appendix). In this case, the set of values of the energy for which we have Liouvillian solutions is given by

$$\left\{ E_n = \frac{|a|}{2n(n+1)} \mid n \in \mathbb{Z} \backslash \{-1, 0\} \right\}.$$

Thus, for $\nu = n$, a basis of solutions of (39) is given by $\xi^{(1)}(t_1) = \sqrt{x_1} J_{n+\frac{1}{2}}(x_1)$ and $\xi^{(2)}(t_1) = \sqrt{x_1} Y_{n+\frac{1}{2}}(x_1)$. Then by Eq. (16) we have $\phi_{12}(t_1)$ as follows:

$$\phi_{12}(t_1) = \frac{t_1}{x_1 - x_0}\sqrt{\frac{x_1}{x_0}}\left(\frac{J_{n+\frac{1}{2}}(x_0)Y_{n+\frac{1}{2}}(x_1) - J_{n+\frac{1}{2}}(x_1)Y_{n+\frac{1}{2}}(x_0)}{J_{n+\frac{1}{2}}(x_0)Y_{n-\frac{1}{2}}(x_0) - J_{n-\frac{1}{2}}(x_0)Y_{n+\frac{1}{2}}(x_0)}\right).$$

Thus, the Van Vleck–Morette determinant reads

$$\det J(t_1, 0) = t_1\phi_{12}(t_1) = \frac{t_1^2}{x_1 - x_0}\sqrt{\frac{x_1}{x_0}}\left(\frac{J_{n+\frac{1}{2}}(x_0)Y_{n+\frac{1}{2}}(x_1) - J_{n+\frac{1}{2}}(x_1)Y_{n+\frac{1}{2}}(x_0)}{J_{n+\frac{1}{2}}(x_0)Y_{n-\frac{1}{2}}(x_0) - J_{n-\frac{1}{2}}(x_0)Y_{n+\frac{1}{2}}(x_0)}\right).$$

Finally, the semiclassical approximation for any value of x_1, x_0, t_1 of the propagator is given by Pauli–Morette formula:

$$K_{\mathrm{WKB}}(\mathbf{x}_1, t_1 \mid \mathbf{x}_0, 0) = \frac{1}{2\pi i\hbar}\frac{\sqrt{x_1 - x_0}}{t_1}\sqrt[4]{\frac{x_0}{x_1}\left(\frac{J_{n+\frac{1}{2}}(x_0)Y_{n-\frac{1}{2}}(x_0) - J_{n-\frac{1}{2}}(x_0)Y_{n+\frac{1}{2}}(x_0)}{J_{n+\frac{1}{2}}(x_0)Y_{n+\frac{1}{2}}(x_1) - J_{n+\frac{1}{2}}(x_1)Y_{n+\frac{1}{2}}(x_0)}\right)}\, e^{\frac{i}{\hbar}\frac{(x_1 - x_0)^2}{2t_1}}.$$

Example. To illustrate the previous case, we consider the Hamiltonian with potential (36) being $\nu = 2$. The expressions for $J_{\frac{5}{2}}$, $Y_{\frac{5}{2}}$, (and so the ones for $J_{\frac{3}{2}}$ and $Y_{\frac{3}{2}}$) read as follows:

$$J_{\frac{5}{2}}(\tau) = -\sqrt{\frac{2}{\pi\tau}}\left(\sin(\tau) + \frac{3\cos(\tau)}{\tau} - \frac{3\sin(\tau)}{\tau^2}\right), \qquad J_{\frac{3}{2}}(\tau) = \sqrt{\frac{2}{\pi\tau}}\left(\frac{\sin(\tau)}{\tau} - \cos(\tau)\right),$$

$$Y_{\frac{5}{2}}(\tau) = \sqrt{\frac{2}{\pi\tau}}\left(\cos(\tau) - \frac{3\sin(\tau)}{\tau} - \frac{3\cos(\tau)}{\tau^2}\right), \qquad Y_{\frac{3}{2}}(\tau) = -\sqrt{\frac{2}{\pi\tau}}\left(\frac{\cos(\tau)}{\tau} + \sin(\tau)\right).$$

Thus, the semiclassical approximation of the propagator is given by Pauli–Morette-formula (4)

$$K_{\mathrm{WKB}}(x_1,t_1 \mid x_0,0) = \frac{1}{2\pi \mathrm{i}\hbar}\,\frac{\sqrt{x_1-x_0}}{t_1}\sqrt[4]{\frac{x_0}{x_1}}\left(\frac{J_{\frac{5}{2}}(x_0)Y_{\frac{3}{2}}(x_0)-J_{\frac{3}{2}}(x_0)Y_{\frac{5}{2}}(x_0)}{J_{\frac{5}{2}}(x_0)Y_{\frac{5}{2}}(x_1)-J_{\frac{5}{2}}(x_1)Y_{\frac{5}{2}}(x_0)}\right)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{(x_1-x_0)^2}{2t_1}}.$$

Having in mind that

$$J_{\frac{5}{2}}(x_0)Y_{\frac{3}{2}}(x_0)-J_{\frac{3}{2}}(x_0)Y_{\frac{5}{2}}(x_0)=\frac{2}{\pi x_0},$$

and setting

$$G := J_{\frac{5}{2}}(x_0)Y_{\frac{5}{2}}(x_1)-J_{\frac{5}{2}}(x_1)Y_{\frac{5}{2}}(x_0),$$

for $0 < x_0 < x_1$, we have

$$G=\frac{(2x_1^2(x_0^2-3)+18x_1x_0-6x_0^2+18)\sin(x_1-x_0)-6(x_1-x_0)(x_1x_0+3)\cos(x_1-x_0)}{\pi(x_1x_0)^{\frac{5}{2}}}.$$

Thus,

$$K_{\mathrm{WKB}}(x_1,t_1 \mid x_0,0)=\frac{1}{\pi^2\mathrm{i}\hbar}\,\frac{\sqrt{x_1-x_0}}{x_0t_1}\sqrt[4]{\frac{x_0}{x_1}}\frac{1}{G}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{(x_1-x_0)^2}{2t_1}}.$$

On the other side, if we set $b = 0$ and $a = 2\nu(\nu+1)E$ in Eq. (38), with $\nu \in \mathbb{R}\backslash\{0,-1\}$, we obtain an Euler–Cauchy differential equation

$$\frac{\partial^2\xi}{\partial\tau^2}=\frac{\nu(\nu+1)}{\tau^2}\,\xi. \tag{40}$$

Its general solution, for $\nu \neq -1/2$, is

$$\xi(\tau)=c_1\xi^{(1)}(\tau)+c_2\xi^{(2)}(\tau)=c_1\tau^{\nu+1}+c_2\tau^{-\nu},\quad \text{with}\quad \xi^{(1)}(\tau)=\tau^{\nu+1},\quad \xi^{(2)}(\tau)=\tau^{-\nu}.$$

As in the previous case, the expression for $\phi_{12}(t_1)$ is derived from Eq. (16):

$$\phi_{12}(t_1)=\frac{x_1^{\nu+1}x_0^{-\nu}-x_0^{\nu+1}x_1^{-\nu}}{(2\nu+1)\sqrt{2E}}=\frac{x_0t_1}{(2\nu+1)(x_1-x_0)}\left(\left(\frac{x_1}{x_0}\right)^{\nu+1}-\left(\frac{x_1}{x_0}\right)^{-\nu}\right).$$

The expression for the propagator K_{WKB} follows analogously:

$$K_{\mathrm{WKB}}(x_1,t_1 \mid x_0,t_0)=\frac{\sqrt{2\nu+1}}{2\pi\mathrm{i}\hbar}\,\frac{\sqrt{\frac{x_1}{x_0}-1}}{t_1\sqrt{\left(\frac{x_1}{x_0}\right)^{\nu+1}-\left(\frac{x_1}{x_0}\right)^{-\nu}}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{(x_1-x_0)^2}{2t_1}}.$$

In the case $\nu = -1/2$ one has

$$\phi_{12}(t_1)=\frac{\sqrt{x_0x_1}}{\sqrt{2E}}(\log(x_1)-\log(x_0))=\frac{\sqrt{x_0x_1}t_1}{x_1-x_0}\log\left(\frac{x_1}{x_0}\right).$$

and the semiclassical approximation of the propagator reads

$$K_{\mathrm{WKB}}(x_1,t_1 \mid x_0,t_0)=\frac{1}{2\pi\mathrm{i}\hbar}\,\frac{\sqrt{x_1-x_0}}{t_1\sqrt[4]{x_0x_1}\sqrt{\log(x_1)-\log(x_0)}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{(x_1-x_0)^2}{2t_1}}.$$

C. Variational equations with Legendre functions

The potentials considered in this section are those of the form

$$V(x,y)=y^2f(x,y)\qquad f(x,y)=-\frac{b}{2}+\frac{a}{2\cosh^2(x)}+\text{h.o.t.}(y). \tag{41}$$

Through the change of variable $\tau = x_E(t) = \sqrt{2E}t + x_0$ in (11), such as in the case of Bessel family, we obtain the variational equation in the form (37).

For a given value of the energy $E > 0$ and consider the free particle motion given by (8). This reduces the study to the one of its variational Eq. (37), which now reads

$$\frac{\partial^2 \xi}{\partial \tau^2} = \frac{1}{2E}\left(b - \frac{a}{\cosh^2(\tau)}\right)\xi. \tag{42}$$

Comparing (42) with the standard differential equation involving the so-called Rosen–Morse potential

$$\frac{\partial^2 \xi}{\partial \tau^2} = \left(\mu^2 - \frac{\nu(\nu+1)}{\cosh^2(\tau)}\right)\xi, \tag{43}$$

we get the conditions

$$\nu(\nu+1) = \frac{a}{2E}, \qquad \mu^2 = \frac{b}{2E},$$

which determine the possible values of the energy. Recall that the Rosen–Morse potentials were studied in Ref. 22 (see also Refs. 23 and 24). To apply differential Galois techniques we introduce the change of variable $z = \tanh(\tau)$ (see also Ref. 24) and obtain

$$(1 - z^2)\frac{\partial^2 \xi}{\partial z^2} - 2z\frac{\partial \xi}{\partial z} + \left(\nu(\nu+1) - \frac{\mu^2}{1 - z^2}\right)\xi = 0. \tag{44}$$

Equation (44) is the well known Legendre equation. Its integrability, in terms of their parameters, is given in Proposition 2 (see the Subsection 2 of the Appendix) and it is only achieved for a discrete set of values of the parameters.

To illustrate the computation of the Feynman propagator for the Legendre family we restrict ourselves to the case $\mu = 0$ (i.e., $b = 0$). The case $b \neq 0$ would follow a similar procedure, using also Proposition 2.

Having in mind Proposition 2(a), the case $\mu = 0$ corresponds to consider $\nu = n \in \mathbb{Z}$. Moreover, without loss of generality, we can assume $n \in \mathbb{N}$. Then, for

$$E_n = \frac{a}{2n(n+1)}, \qquad \mu_n^2 = \frac{b}{2E_n} \quad n \in \mathbb{N},$$

the general solution of Legendre Eq. (44) is given by

$$\xi(z) = c_1 P_n(z) + c_2 Q_n(z),$$

where $P_n(z)$ denotes the Legendre polynomial of degree n and $Q_n(z)$ an independent solution determined by D'Alembert formula. Therefore, its general solution becomes

$$\xi(\tau) = c_1 P_n(\tanh(\tau)) + c_2\ Q_n(\tanh(\tau)), \qquad \tau = \frac{x_1 - x_0}{t_1}t + x_0.$$

That is, $\{\xi^{(1)}(\tau), \xi^{(2)}(\tau)\} = \{P_n(\tanh(\tau)), Q_n(\tanh(\tau))\}$ is a basis of solutions. Thus,

$$\xi^{(1)}(0) = P_n(\tanh(x_0)), \qquad \xi^{(2)}(0) = Q_n(\tanh(x_0)), \qquad \xi^{(1)}(t_1) = P_n(\tanh(x_1)), \qquad \xi^{(2)}(t_1) = Q_n(\tanh(x_1))$$

and

$$\dot{\xi}^{(1)}(t=0) = \frac{x_1 - x_0}{t_1\cosh^2(x_0)}P_n'(\tanh(x_0)), \qquad \dot{\xi}^{(2)}(t=0) = \frac{x_1 - x_0}{t_1\cosh^2(x_0)}Q_n'(\tanh(x_0)),$$

where $'$ denotes, in this case, derivative with respect to τ. Applying formula (16) we obtain

$$\phi_{12}(t_1) = \frac{t_1\cosh^2(x_0)}{x_1 - x_0}\cdot\frac{P_n(\tanh(x_0))\ Q_n(\tanh(x_1)) - P_n(\tanh(x_1))\ Q_n(\tanh(x_0))}{P_n(\tanh(x_0))\ Q_n'(\tanh(x_0)) - P_n'(\tanh(x_0))\ Q_n(\tanh(x_0))},$$

and the corresponding semiclassical approximation of the propagator is

$$K_{\mathrm{WKB}}(x_1, t_1 \mid x_0, 0) = \frac{1}{2\pi i\hbar}\ \frac{1}{\sqrt{t_1\phi_{12}(t_1)}}\ e^{\frac{i}{2\hbar t_1}(x_1 - x_0)^2}.$$

D. Variational equations with Lamé functions

This family corresponds to potentials of type

$$V(x,y) = y^2 f(x,y) \quad f(x,y) = \left(-\frac{b}{2} - \frac{a}{2}\wp(x+\omega_3)\right) + \text{h.o.t.}(y), \tag{45}$$

where, on the free particle solution, it simply reads

$$f(x_E(t),0) = -\frac{b}{2} - \frac{a}{2}\wp(x_E(t)+\omega_3),$$

with $x_E(t) = \sqrt{2E}\, t + x_0 = \frac{x_1 - x_0}{t_1} t + x_0$. Notice that the potential (45) depends on four parameters: a, b, g_2 and g_3; the parameter ω_3 depending, on its turn, on g_2 and g_3, see Subsection 3 of the Appendix and for more details see Ref. 25.

By the change $\tau = x_E(t) + \omega_3$, the variational Eq. (11) becomes the celebrated Lamé differential equation

$$\frac{d^2\xi}{d\tau^2} = (n(n+1)\wp(\tau) + B))\xi, \tag{46}$$

with

$$n(n+1) = \frac{a}{2E}, \qquad B = \frac{b}{2E}.$$

According to the Subsection 3 of the Appendix, the three cases of differential Galois integrability of Lamé equation lead only to a discrete set of values of parameters. Indeed,

$$E = E_n = \frac{a}{2n(n+1)}, \qquad B = B_n = n(n+1)\frac{b}{a}.$$

As we did for the Legendre family of potentials, we illustrate the theory by considering a concrete example. Namely, let us take $n = 1$ and $a = 2$. Hence, we our potential becomes

$$V(x,y) = y^2 f(x,y), \qquad f(x,y) = \left(-\frac{b}{2} - \wp(x+\omega_3)\right) + \text{h.o.t.}(y). \tag{47}$$

We recall that $\wp$ is the Weierstrass function, which is a solution of the differential equation $(dw/dz)^2 = h(w)$, with $h(w) = 4w^3 - g_2 w - g_3$ and discriminant $\Delta = g_2^3 - 27g_3^2 \neq 0$ in order that the polynomial $h(w)$ has simple roots (otherwise it can be transformed into simpler forms).

Here, we consider $h(B) \neq 0$, which falls into the Hermite–Halphen family, an integrable case of the Lamé equation (see Subsection 3 of the Appendix and also Refs. 25 and 26).

Example. Fix $b = 1$ in the potential (47) and $h(w) = 4w^3 - 28w + 24$, with roots $e_1 = 2$, $e_2 = 1$, and $e_3 = -3$. The parameters are $g_2 = 28$, $g_3 = -24$ and (according to the notation of Subsection 3 of the Appendix) the discriminant is $\Delta = 6400$. To ease the computations we fix $x_0 = 0$. We have $E_1 = 1/2$ and define $\tau = t + \omega_3$. Moreover, $h(B) \neq 0$, i.e., $B \neq 1, 2, -3$. Then Eq. (46) becomes

$$\ddot{\xi} = (B + 2\wp(t))\xi, \tag{48}$$

with a basis of solutions $\{\xi^{(1)}(t), \xi^{(2)}(t)\}$, where

$$\xi^{(1)}(t) = \sqrt{B - \wp(t+\omega_3)}\; e^{\frac{1}{2}\sqrt{h(B)}\int_0^t \frac{ds}{B-\wp(s+\omega_3)}}, \qquad \xi^{(2)}(t) = \sqrt{B - \wp(t+\omega_3)}\; e^{-\frac{1}{2}\sqrt{h(B)}\int_0^t \frac{ds}{B-\wp(s+\omega_3)}}.$$

For $t = 0$, we have $\tau = \omega_3$ and therefore $\xi^{(1)}(0) = \sqrt{B+3}$, $\xi^{(2)}(0) = \sqrt{B+3}$. Thus, the value of D in formula (16) is $D = 2\sqrt{h(B)}$ and

$$\phi_{12}(t_1) = \sqrt{\frac{(B+3)(B - \wp(t_1+\omega_3))}{h(B)}}\; \sinh\left(\frac{1}{2}\sqrt{h(B)}\int_0^{t_1} \frac{ds}{B - \wp(s+\omega_3)}\right).$$

Finally,

$$K_{\mathrm{WKB}}(x_1, t_1 \mid x_0, 0) = \frac{1}{2\pi i\hbar}\, \frac{1}{\sqrt{t_1 \phi_{12}(t_1)}}\; e^{\frac{i}{2\hbar t_1}(x_1 - x_0)^2}.$$

IV. FINAL REMARKS: NON-INTEGRABILITY AND FUTURE WORK

In this section we precise the non-integrability statements concerning the families of potentials and the corresponding propagators considered in this paper and we present some questions for future work.

In fact, the families considering here are generically non-integrable.

Proposition 1. Consider the families of potentials defined by Eq. (5) *with* $k = 2$ *and given in Table* (17).

Then, the corresponding Hamiltonian systems whose variational equations reduce to Bessel and Legendre families with $b \neq 0$ *and those which reduce to the Hermite and Lamé families are not integrable in the Liouville–Arnold sense.*

Proof. The main common argument is the following: according to Sec. III, the variational equations of the Hamiltonian systems are differential Galois integrable only for discrete sets of the energy. Consequently, the Hamiltonian systems cannot be integrable in the Liouville–Arnold sense.

First, consider the Hermite and Bessel families. As there are values of the energy E for which the corresponding variational equations are not integrable in the sense of differential Galois theory, then the Hermite and Bessel families of Hamiltonian systems are not integrable in the Liouville–Arnold sense (see Refs. 12 and 27).

Now we consider Legendre family. From Proposition 2 (see Subsection 2 of the Appendix) we notice that only discrete values of the energy are compatible with integrability. Therefore the main argument above applies.

And last, but not least, we consider the Lamé family. For the case of Lamé and the Hermite–Halphen solutions, the Brioschi–Halphen–Crawford solutions as well as the Baldassarri solutions, they are only compatible with integrability for discrete values of the energy (for more details see Ref. 26, Sec. 3 and also Ref. 12). Analogously, the claim follows. ■

We stress that, in Proposition 1, since for the Hermite and Bessel families the variational equations have irregular singular points at infinity, then the obstruction is to the existence of an additional rational first integral. For the other two families, the obstruction is to the existence of an additional meromorphic first integral (see Refs. 12 and 27). In the philosophy of the papers,[1,13] we introduce the following definition:

Definition 1. The semiclassical approximation of the Feynman propagator is integrable if for any fixed classical path γ *the expression* $K_{\mathrm{WKB}} = K_{\mathrm{WKB}}(t_1)$ *is a Liouvillian function over the coefficient field of the associated variational equation.*

In the above definition, the points x_0 and x_1 are considered fixed and so, the semiclassical aproximation of the Feynman propagator (4) becomes a function depending only on time t_1.

If there exists a classical path whose $K_{\mathrm{WKB}} = K_{\mathrm{WKB}}(t)$ is not a Liouvillian function, then the semiclassical approximation of the Feynman propagator is not integrable.

Recall that a Liouvillian function over a differential field of functions $\mathcal{K}$ is a function obtained by a combination of algebraic functions, integrals and exponential of integrals of functions in $\mathcal{K}$. Then, the solutions of a linear differential equation over $\mathcal{K}$ is given by Liouvillian functions if, and only if, the linear differential equation is integrable (for a more formal statement see the Appendix in Ref. 1). The differential coefficient field $\mathcal{K}$ along this paper is generated by the function $f(x_E(t), 0)$: see formula (10). It is clear from formula (14) that $K_{\mathrm{WKB}}(t_1)$ is a Liouvillian function if, and only if, $\phi_{12}(t_1)$ is, on its turn, a Liouvillian function. But, by (16), $\phi_{12}(t_1)$ is given as a linear combination of the base of solutions of the normal variational equation. Consequently, all the closed form formulas of $\phi_{12}(t_1)$ obtained in this paper are given by Liouvillian functions.

However, from the proof of the above proposition, it becomes clear that it will not be possible to obtain for some of the paths a base of solutions of the normal variational equation as Liouvillian functions. So, the following corollary can be stated:

Corollary 1. Consider the families of potentials defined by Eq. (5) *with* $k = 2$ *given in Table* (17). *Then the semiclassical approximations of the Feynman propagator of Bessel and Legendre families with* $b \neq 0$*, as well as Hermite and Lamé families, are not integrable.*

Remark 1 (This remark is motivated by an observation of the anonymous referee, to whom we are grateful). The restriction to discrete values of the energy E_n *to obtain solutions of the propagator in closed form seems to be related to the existence of a discrete spectrum for the normal fluctuating operator which defines the normal variational equation. Similarly, it might also be linked to the connection between the Van Vleck (or Van Vleck–Morette) determinants and their associated functional determinants via Gelfand–Yaglom's approach. This seems to be the case for the Hermite (Sec. III A) and Legendre (Sec. III C) families. However, it is not so clear for us in the case of Bessel family (Sec. III B). Regarding the Lamé family it should be necessary to approach the spectral problem as a Floquet periodic problem, where the eigenfunctions, the Lamé solutions, are obtained for the periodic and the anti-periodic spectrum. We believe that a detailed analysis of these facts fall outside the target of this paper.*

We certainly believe that there are other, less academic, more challenging applications of this theory to physical problems. For instance, those related to the quantification of periodic orbits or the ones regarding *tunneling*. More precisely, three of the problems we would like to address in future works are:

1. The quantification around families of periodic orbits parametrized by the energy, following essentially the work of Gutzwiller.[28]
2. The macroscopic tunneling problems in magnetism, as proposed in the book of Chudnovsky and Tejada.[29] However most of the systems appearing therein are one-degree of freedom, the use of a non-standard Hamiltonian function (that is, not of the form kinetic + potential) can give rise to interesting dynamical features.
3. Tunneling problems in higher degrees of freedom systems, but with enough number of symmetries.

ACKNOWLEDGMENTS

P.A.-H. is partially supported by the Ministerio de Educación Superior, Ciencia y Tecnología (MESCyT) Grant No. FONDOCYT 2022-1D2-091 "Aproximación Semiclásica de Sistemas Hamiltonianos con potenciales Homogéneos de dos grados de libertad y osciladores no autónomos."

J.T.L. has been supported by the Spanish State Research Agency (AEI), through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R & D (Grant No. CEX2020-001084-M). Moreover, he has been funded by the Spanish Project No. PGC2018-098676-B-100 funded by MCIN/AEI/10.13039/501100011033 "ERDF A way of making Europe;" by the Spanish Project No. PID2021-122954NB-I00 funded by MCIN/AEI/10.13039/501100011033/ and "ERDF A way of making Europe;" and by a grant from the "Ayudas para la recualificación del sistema universitario español para 2021–2023." He also thanks Laboratorio Subterráneo de Canfranc (LSC), the Instituto de Biología Integrativa de Sistemas (I2Sysbio, CSIC-UV), and the Institut de Mathématiques de Jussieu - Paris Rive Gauche (Sorbonne Université) for their kind hospitality as the hosting institutions of this grant.

J.J.M.R. has been supported by the Universidad Politécnica de Madrid research group *Modelos Matemáticos no lineales*. As well as he thanks to the Japan Society for Promotion of the Science and to the hospitality of the Departament of Applied Mathematics and Physics of Kyoto University, where he stayed in the fall of 2022 under the Grant "FY JSPS Invitational Fellowships for Research in Japan" and where part of his contribution to this work was done.

C.P. is partially supported by the Ministerio de Ciencia e Innovación Grant (No. PID2019-104658GB-I00); is funded partially by the Grant No. PID-2021-122954NB-100 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe."

P.A.-H. and J.J.M.R. thanks EPSEB (UPC) for its hospitality during their visit at the final stages of this work.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: BESSEL, LEGENDRE AND LAMÉ EQUATIONS

We provide some additional theoretical background to make the paper more self-contained. They aim to complement the Applications Sec. III.

1. Bessel equation

The well-known Bessel differential equation (see, for instance, Ref. 21) is a second order linear ode of type:

$$x^2\frac{d^2z}{dx^2}+x\frac{dz}{dx}+\left(x^2-\alpha^2\right)z=0,\qquad \alpha\in\mathbb{C}. \tag{A1}$$

It is well known that for values of the parameter $\alpha \in \mathbb{Z} + \frac{1}{2}$ (i.e., α a half-integer) this equation is integrable in the Picard–Vessiot sense, that is, it admits Liouvillian solutions (see Ref. 12 and references therein). Let us assume that we are in such case, and so $\alpha \in \mathbb{Z} + \frac{1}{2}$. The general solution of (A1) can be expressed as

$$z(x) = c_1 J_\alpha(x) + c_2 Y_\alpha(x),$$

where J_α and Y_α are the (so-called) Bessel functions of first and second kind of parameter α, respectively (see Ref. 21). The change of variables $y = \sqrt{x}z$ (see Ref. 30) brings the Bessel equation (A1) into its normal form, namely,

$$y'' = r(x)y, \quad r(x) = \frac{\alpha^2 - \frac{1}{4}}{x^2} - 1, \tag{A2}$$

where $'$ denotes differentiation with respect to x. Moreover, since $\alpha \in \mathbb{Z} + \frac{1}{2}$ we can set $\alpha = n + \frac{1}{2}$, with $n \in \mathbb{Z}$. Thus, $r(x)$ above becomes

$$r(x) = \frac{\left(\alpha - \frac{1}{2}\right)\left(\alpha + \frac{1}{2}\right)}{x^2} - 1 = \frac{n(n+1)}{x^2} - 1.$$

Finally, the transformation $x \mapsto \mu x$, with $\mu \neq 0$, leads Eq. (A2) into

$$y'' = \left(\frac{n(n+1)}{x^2} - \mu^2\right)y, \quad n \in \mathbb{Z},\ \mu \in \mathbb{C}^*. \tag{A3}$$

The case $n = 0, -1$ corresponds to a differential equation with constant coefficients, so clearly integrable.

Equation (A3) is very well known in mathematical physics because it corresponds to a radial free particle the Schrödinger equation.

2. Legendre equation

The general Legendre ordinary differential equation takes the form

$$(1 - z^2)\frac{\partial \xi}{\partial z^2} - 2z\frac{\partial \xi}{\partial z} + \left(n(n+1) - \frac{m^2}{1 - z^2}\right)\xi = 0. \tag{A4}$$

It has regular singularities at the points $z = -1, 1, \infty$. Both singular points $z = \pm 1$ have indices $m/2, -m/2$, while the singular point $z = \infty$ has indices $-n, n+1$. Hence the differences between the indices at the points $z = -1, 1, \infty$ are, up to change in sign, $(m, m, -2n - 1)$, respectively. The following Proposition states the values of the parameters m, n for which it is integrable. It is a correction, completing some missing cases, of Ref. 31, Proposition 4, and it is a direct consequence of Kimura's Theorem.[32]

Proposition 2. The Legendre equation (A4) is integrable if and only if, either one of the following cases holds:

(A) *exactly one of the following situations is satisfied*
   1. $n \in \mathbb{Z}$
   2. $m + n \in \mathbb{Z}$, $m \notin \mathbb{Z}$ *and* $n \notin \mathbb{Z}$
   3. $m - n \in \mathbb{Z}$ $m \notin \mathbb{Z}$ *and* $n \notin \mathbb{Z}$.

(B) *m, n belong to one of the following families of cases:*

Case	$m \in$	$n \in$
1	$\frac{1}{2}(2\mathbb{Z}+1)$	$\mathbb{C}$
2	$\frac{1}{3}(3\mathbb{Z}\pm 1)$	$\frac{1}{4}(2\mathbb{Z}+1)$
3	$\frac{1}{3}(3\mathbb{Z}\pm 1)$	$\frac{1}{6}(6\mathbb{Z}\pm 1)$
4	$\frac{1}{4}(4\mathbb{Z}\pm 1)$	$\frac{1}{6}(6\mathbb{Z}\pm 1)$
5	$\frac{1}{3}(3\mathbb{Z}\pm 1)$	$\frac{1}{10}(10\mathbb{Z}\pm 3)$
6	$\frac{1}{5}(5\mathbb{Z}\pm 1)$	$\frac{1}{6}(6\mathbb{Z}\pm 1)$
7	$\frac{1}{5}(5\mathbb{Z}\pm 2)$	$\frac{1}{10}(10\mathbb{Z}\pm 3)$
8	$\frac{1}{5}(5\mathbb{Z}\pm 1)$	$\frac{1}{10}(10\mathbb{Z}\pm 1)$

Let us deal with the proof of this proposition. We denote by $\lambda = m$, $\mu = m$ and $\nu = -(2n+1)$ the exponent differences and we analyze all the cases of Kimura's Theorem.[32] Hence, in order the Legendre Eq. (44) to have Liouvillian solutions it is necessary and sufficient that, either case A or case B holds. Let us proceed case by case.

Case A. At least one of $\lambda+\mu+\nu$, $-\lambda+\mu+\nu$, $\lambda-\mu+\nu$ or $\lambda+\mu-\nu$ is an odd integer. Equivalently, at least one of the following relations is satisfied: $\lambda+\mu+\nu \in 2\mathbb{Z}+1$, $-\lambda+\mu+\nu \in 2\mathbb{Z}+1$, $\lambda-\mu+\nu \in 2\mathbb{Z}+1$ or $\lambda+\mu-\nu \in 2\mathbb{Z}+1$.

We consider each item separately.

1.1 Relation $\lambda+\mu+\nu = 2m-2n-1 \in 2\mathbb{Z}+1$ yields to $m-n \in \mathbb{Z}$. In conclusion $(m,n) \in \mathbb{Z}^2$ or $m \notin \mathbb{Z}$ and $n \notin \mathbb{Z}$ with $m-n \in \mathbb{Z}$.
1.2 Relation $-\lambda+\mu+\nu = 2n-1 \in 2\mathbb{Z}+1$ yields to $n \in \mathbb{Z}$. In conclusion $n \in \mathbb{Z}$ and $m \in \mathbb{C}$.
1.3 Relation $\lambda-\mu+\nu = 2n-1 \in 2\mathbb{Z}+1$ yields to $n \in \mathbb{Z}$. In conclusion $n \in \mathbb{Z}$ and $m \in \mathbb{C}$.
1.4 Relation $\lambda+\mu-\nu = 2m+2n+1 \in 2\mathbb{Z}+1$ yields to $m+n \in \mathbb{Z}$. In conclusion $(m,n) \in \mathbb{Z}^2$ or $m \notin \mathbb{Z}$ and $n \notin \mathbb{Z}$ with $m+n \in \mathbb{Z}$.
1.5 Any combination of the previous four relations does not provide any new condition on the parameters n and m.

In this way, the first part of Kimura's Theorem for Legendre differential Eq. (44) is proved and, summarising, we obtain:

1. $n \in \mathbb{Z}$;
2. $m+n \in \mathbb{Z}$, $m \notin \mathbb{Z}$ and $n \notin \mathbb{Z}$;
3. $m-n \in \mathbb{Z}$ $m \notin \mathbb{Z}$ and $n \notin \mathbb{Z}$;

Case B. The quantities λ or $-\lambda$, μ or $-\mu$, ν or $-\nu$ take, in an arbitrary order, values given in Kimura's table, see Ref. 32. The cases 4, 6, 9, 10, 12, 14 and 15 of Kimura's table are discarded because for Legendre Eq. (44) we have that two differences of exponents have the value.

Now we check the rest of the cases of Kimura's table:

Case 1. By case 1 in Kimura's table, we have $\lambda = \mu = m \in \pm(\mathbb{Z}+\frac{1}{2}) = (\mathbb{Z}\pm\frac{1}{2}) = \frac{1}{2}(2\mathbb{Z}\pm 1)$ and $\nu = -(2n+1) \in \mathbb{C}$, which lead us to $m \in \frac{1}{2}(2\mathbb{Z}+1)$ and $n \in \mathbb{C}$. Thus, we obtain the case 1 in the table provided in Proposition 2.

Case 2. By case 2 in Kimura's table the only one possibility for m and n is provided by $\lambda = \mu = m \in \pm(\mathbb{Z}+\frac{1}{3}) = (\mathbb{Z}\pm\frac{1}{3}) = \frac{1}{3}(3\mathbb{Z}\pm 1)$ and $\nu = -(2n+1) \in \pm(\mathbb{Z}+\frac{1}{2}) = (\mathbb{Z}\pm\frac{1}{2}) = \frac{1}{2}(2\mathbb{Z}\pm 1)$, which lead us to $m \in \frac{1}{3}(3\mathbb{Z}\pm 1)$ and $n \in \frac{1}{4}(2\mathbb{Z}+1)$. Thus, we obtain the case 2 in the table provided in Proposition (2).

Case 3. By case 3 in Kimura's table the only possibility for m and n is provided by $\lambda = \mu = m \in \pm(\mathbb{Z}+\frac{1}{3}) = (\mathbb{Z}\pm\frac{1}{3}) = \frac{1}{3}(3\mathbb{Z}\pm 1)$ and $\nu = -(2n+1) \in \pm(2\mathbb{Z}+\frac{2}{3}) = (2\mathbb{Z}\pm\frac{2}{3})$. Therefore $n+\frac{1}{2} \in \mathbb{Z}\pm\frac{1}{3}$, which leads to $m \in \frac{1}{3}(3\mathbb{Z}\pm 1)$ and $n \in \frac{1}{6}(6\mathbb{Z}\pm 1)$. Thus, we obtain the case 3 at the table provided in the statement of the Proposition.

Case 5. By case 5 in Kimura's table and due to the even condition the only possibility for m and n is provided by $\lambda = \mu = m \in \pm(\mathbb{Z}+\frac{1}{4}) = (\mathbb{Z}\pm\frac{1}{4}) = \frac{1}{4}(4\mathbb{Z}\pm 1)$ and $\nu = -(2n+1) \in \pm(2\mathbb{Z}+\frac{2}{3}) = (2\mathbb{Z}\pm\frac{2}{3})$. Therefore $n+\frac{1}{2} \in \mathbb{Z}\pm\frac{1}{3}$, which lead us to $m \in \frac{1}{4}(4\mathbb{Z}\pm 1)$ and $n \in \frac{1}{6}(6\mathbb{Z}\pm 1)$. Thus, we obtain the case 4 in the table of the statement.

Case 7. By case 7 in Kimura's table and due to the even condition the only possibility for m and n is provided by $\lambda = \mu = m \in \pm(\mathbb{Z}+\frac{1}{3}) = (\mathbb{Z}\pm\frac{1}{3}) = \frac{1}{3}(3\mathbb{Z}\pm 1)$ and $\nu = -(2n+1) \in \pm(2\mathbb{Z}+\frac{2}{5}) = (2\mathbb{Z}\pm\frac{2}{5})$. Therefore $n+\frac{1}{2} \in \mathbb{Z}\pm\frac{1}{5}$, which lead to $m \in \frac{1}{3}(3\mathbb{Z}\pm 1)$ and $n \in \frac{1}{10}(10\mathbb{Z}\pm 3)$. Thus, we obtain the case 5 in the table provided in Proposition 2.

Case 8. By case 8 in Kimura's table and due to the even condition the only possibility for m and n is provided by $\lambda = \mu = m \in \pm(\mathbb{Z} + \frac{1}{5}) = (\mathbb{Z} \pm \frac{1}{5}) = \frac{1}{5}(5\mathbb{Z} \pm 1)$ and $\nu = -(2n+1) \in \pm(2\mathbb{Z} + \frac{2}{3}) = (2\mathbb{Z} \pm \frac{2}{3})$. Therefore $n + \frac{1}{2} \in \mathbb{Z} \pm \frac{1}{3}$, which leads to $m \in \frac{1}{5}(5\mathbb{Z} \pm 1)$ and $n \in \frac{1}{6}(6\mathbb{Z} \pm 1)$. Thus, we obtain the case 6 in the table provided in the Proposition.

Case 11. By case 11 in Kimura's table and due to the even condition the only possibility for m and n is provided by $\lambda = \mu = m \in \pm(\mathbb{Z} + \frac{2}{5}) = (\mathbb{Z} \pm \frac{2}{5}) = \frac{1}{5}(5\mathbb{Z} \pm 2)$ and $\nu = -(2n+1) \in \pm(2\mathbb{Z} + \frac{2}{5}) = (2\mathbb{Z} \pm \frac{2}{5})$. Therefore $n + \frac{1}{2} \in \mathbb{Z} \pm \frac{1}{5}$, which leads to $m \in \frac{1}{5}(5\mathbb{Z} \pm 2)$ and $n \in \frac{1}{10}(10\mathbb{Z} \pm 3)$. Thus, we obtain the case 7 in the table.

Case 13. By case 13 in Kimura's table and due to the even condition the only possibility for m and n is provided by $\lambda = \mu = m \in \pm(\mathbb{Z} + \frac{1}{5}) = (\mathbb{Z} \pm \frac{1}{5}) = \frac{1}{5}(5\mathbb{Z} \pm 1)$ and $\nu = -(2n+1) \in \pm(2\mathbb{Z} + \frac{4}{5}) = (2\mathbb{Z} \pm \frac{4}{5})$. Therefore $n + \frac{1}{2} \in \mathbb{Z} \pm \frac{2}{5}$, leading to $m \in \frac{1}{5}(5\mathbb{Z} \pm 1)$ and $n \in \frac{1}{10}(10\mathbb{Z} \pm 1)$. Thus, we obtain the case 8 in the table.

3. Lamé equation

The well-known Lamé ordinary differential equation is, in Weierstrass form, given by

$$\frac{d^2y}{dz^2} = (n(n+1)\wp(z) + B)y, \tag{A5}$$

where $\wp$ is the Weierstrass function, a solution of the differential equation $(dw/dz)^2 = h(w)$, with $h(w) = 4w^3 - g_2 w - g_3$ and discriminant $\Delta = g_2^3 - 27g_3^2 \neq 0$ [in order the polynomial $h(w)$ to have simple roots; otherwise it could be transformed into a simpler form]. This Eq. (A5) depends on four parameters: n, B, g_2 and g_3.

We assume the basic periods of $\wp$, named $2\omega_1, 2\omega_3$, to be real and purely imaginary, respectively. These conditions are satisfied when g_2 and g_3 are real and $\Delta > 0$. If we denote by e_1, e_2 and e_3 the roots of $h(w)$, we have that they are all real and it is no restrictive to assume that $e_3 < e_2 < e_1$. Then $\omega_i = e_i$, $i = 1, 2, 3$, being ω_1 real and ω_3 purely imaginary. The function $g(x) = \wp(x + \omega_3)$ is real and regular for $x \in \mathbb{R}$ (see for instance Ref. 25).

Some integrability conditions of Lamé equation (A5), in the sense of the differential Galois theory, can be found in Refs. 12, 25, and 26. They are:

(i) *Lamé and Hermite–Halphen solutions*, $n \in \mathbb{N}$.
(ii) *Brioschi–Halphen–Crawford solutions*, $n + 1/2 \in \mathbb{N}$ and some algebraic conditions on the rest of the parameters.
(iii) *Baldasarri solutions*, $n + \frac{1}{2} \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z} \setminus \mathbb{Z}$ and some other involved conditions on the rest of parameters.

In particular, for $n \in \mathbb{N}$, Lamé equation becomes integrable.

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