Exercises

1. Apply the Sardinas-Patterson theorem to determine which of the following codes are uniquely decodable. In the case of non-uniquely decodable give a sequence which can be decoded in two different ways.

   i. $\{01, 0011, 2, 102, 0120\}$

   ii. $\{0100, 0011, 2, 102, 2012, 2001, 0220, 12102\}$

2. By using Kraft's inequality determine (if possible) prefix codes with the following word lengths.

   i. $\{1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4\}$ over a 3 symbol alphabet.

   ii. $\{1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 4\}$ over a 4 symbol alphabet.

3. Construct the binary Huffman code for a source with probabilities

$$0.3, 0.25, 0.15, 0.15, 0.05, 0.05, 0.05,$$

   and calculate the average word length.

4. Let C be the Huffman code of a source S, where the two (of the) least likely symbols to be emitted from S are s_{q-1} and s_q. Let C' be the Huffman code of the source $S' = (S \setminus \{s_{q-1}, s_q\}) \cup \{s'\}$, where s' is emitted with probability $p_{q-1} + p_q$. Prove that $L(C) - L(C') = p_{q-1} + p_q$.

5. Calculate the average word length of the binary Shannon-Fano code and the binary Huffman code for the source S with probabilities $0.4, 0.25, 0.15, 0.1, 0.1$. Compare these with the entropy $H_2(S)$.

6. Calculate the average word length of the Shannon-Fano code of S^n, where S is a two symbol source with probabilities $\frac{3}{4}$ and $\frac{1}{4}$, and verify the noiseless coding theorem for this source.

7. Prove that for the repitition code $C = \{11 \ldots 11, 22 \ldots 22, \ldots, rr \ldots rr\} \subset \{1, 2, \ldots, r\}^n$ and using a binary symmetric channel $P_{COR} \to 1$ and $R \to 0$ as $n \to \infty$.

8. Show that for $q = 2^h$, the matrix whose columns are $\{(1, t, t^2) \mid t \in \mathbb{F}_q\} \cup \{(0, 1, 0), (0, 0, 1)\}$ generates a 3-dimensional maximum distance separable of length $q + 2$ over $\mathbb{F}_q$. Hint: Use that $x = -x$ for all $x \in \mathbb{F}_q$.

9. Prove that the dual of a linear maximum distance separable code is a maximum distance separable code.

10. Let C be the linear code over $\mathbb{F}_5$ generated by the matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}.$$

Calculate a parity check matrix for C, the minimum distance of C, and decode the received vector $(0, 2, 3, 4, 3, 2)$.

11. Compute the parameters of all Hamming codes that are also MDS. Build the ternary Hamming code of length 4 and show that it is selfdual.