Codes and Cryptography

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PART VIII

Outline

Why Looking for More Cyclic Codes?

Reed-Solomon codes can be seen as MDS cyclic codes. But over a fixed field $\mathbb{F}_q$, MDS codes of arbitrary length $n > q + 2$, are conjectured to exist only when $k = 1$ (the repetition code) or $k \geq n - 1$ (that is $d = 1$ or 2).

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E.g., BCH codes.

BCH Codes

Let $\beta \in \mathbb{F}_{q^e}$ be a n-th primitive root of unity, for a minimal e.
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For some integer $1 \leq m \leq n-1$ consider the minimal polynomial $g \in \mathbb{F}_q[X]$ of $\beta, \beta^2, \ldots, \beta^m$, that is the monic polynomial of minimal degree vanishing at $\beta, \beta^2, \ldots, \beta^m$.

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If $e = 1$, then $n \mid q - 1$, $\deg g = m$, $d = m + 1$ and $\mathcal{C}$ is MDS.

Proof of the lower bound $d \geq m + 1$

Assume that there is a codeword of weight m (or less)
$w = \lambda_1 X^{i_1} + \ldots + \lambda_m X^{i_m}$, where $i_1 < \cdots < i_m$

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Then, $w \in \mathcal{C}$ implies that $w(\beta^j) = 0$ for $j = 1, \ldots, m$.
This means that $(\lambda_1, \ldots, \lambda_m)$ is a solution to

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The same proof works for any set of roots of the form $\beta^a, \beta^{a+b}, \ldots, \beta^{a+bm}$ for $\gcd(b, n) = 1$.

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Conjugate classes: $\{1\}$ $\{\beta, \beta^2, \beta^4, \beta^8, \beta^{16}, \beta^{15}, \beta^{13}, \beta^9\}$
$\{\beta^3, \beta^6, \beta^{12}, \beta^7, \beta^{14}, \beta^{11}, \beta^5, \beta^{10}\}$

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Taking $m = 2$: (The **design distance** is $d^* = m + 1 = 3$)
$g = (X - \beta)(X - \beta^2)(X - \beta^4)(X - \beta^8)(X - \beta^{16})(X - \beta^{15})(X - \beta^{13})(X - \beta^9) =$
$= X^8 + X^7 + X^6 + X^4 + X^2 + X + 1$
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$\mathcal{C}$ is a $[17, 9, 5]_2$ BCH code, with an optimal dimension.

The sphere packing bound reads here $k \leq \left\lfloor 17 - \log_2(1 + 17 + \binom{17}{2}) \right\rfloor = 9$

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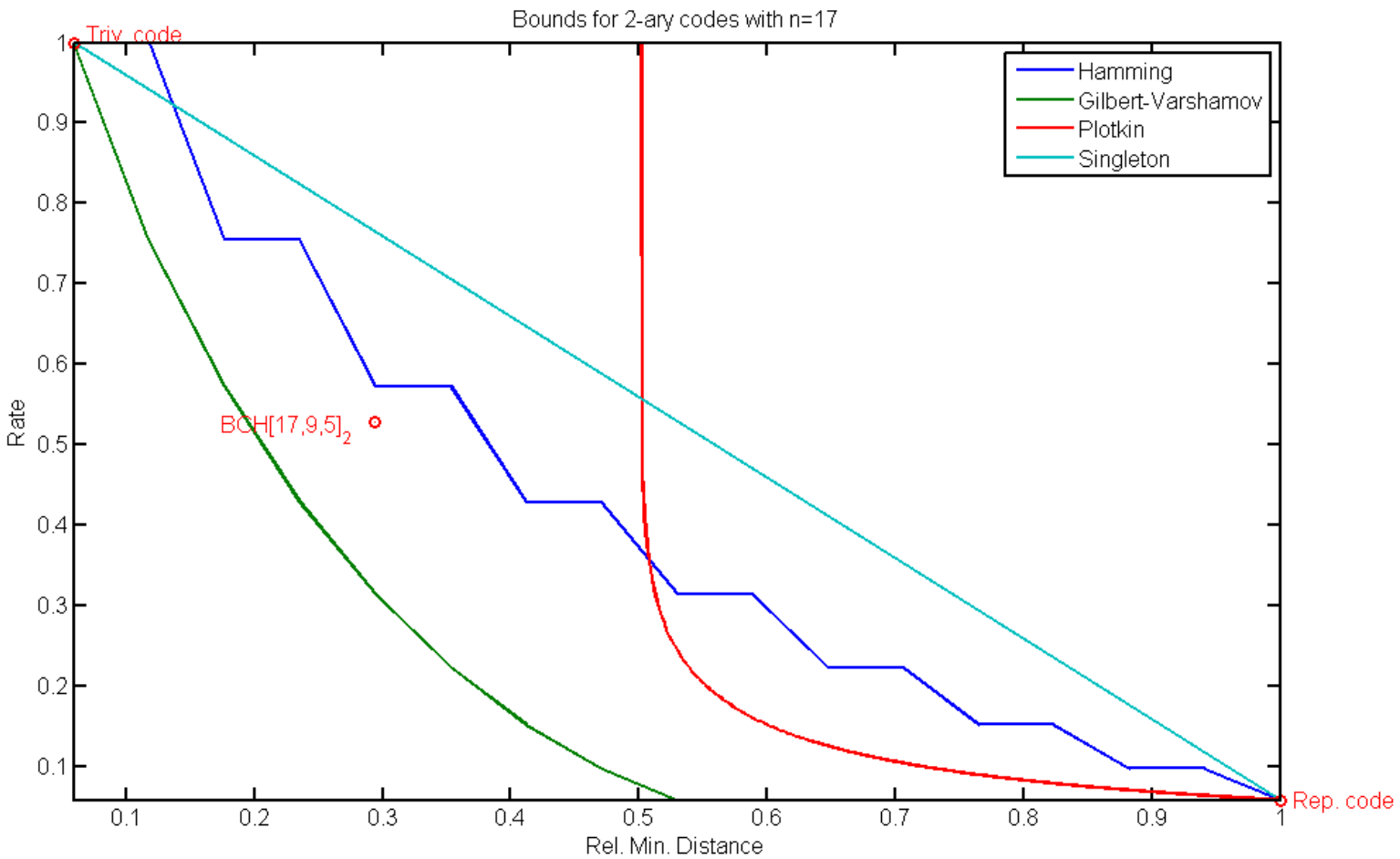

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- $\mathcal{C}'$ is an MDS code,
- encoding in $\mathcal{C}'$ can be computed by division by g,
- error correction in $\mathcal{C}'$ works the same way as in the original cyclic Reed-Solomon code $\mathcal{C}$.

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- Errors tend to follow some predefined timing pattern. Because symbols are stored into a two dimensional structure, and there are new adjacency relations.

This makes errors accumulate in a single codeword, typically exceeding the error correction capability of the code.

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Build a complex code from two simpler ones by composition of the encoding functions. Then apply the two layers of error correction.

The resulting encoding function is simpler than the encoding function of a directly designed large code.

Correcting a large number of errors is also more efficient.

Adds protection against some specific error patterns.

Concatenation of Linear Codes (I)

Let $\mathcal{C}_1 \subset (\mathbb{F}_{q_1})^{n_1}$ be a $[n_1, k_1, d_1]_{q_1}$-linear code, called the **inner code**, and let $\mathcal{C}_2 \subset (\mathbb{F}_{q_2})^{n_2}$ be a $[n_2, k_2, d_2]_{q_2}$-linear code, called the **outer code**.

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We assume the existence of a q-linear bijection $\psi : \mathbb{F}_{q_2}^{n_2} \to \mathbb{F}_{q_1}^{k_1}$
Thus, $q_1 = q^{e_1}$, $q_2 = q^{e_2}$ for some q, e_1, e_2 and $e_1 k_1 = e_2 n_2$.
We also use natural q-linear bijections $\mathbb{F}_{q^{e_1}} \cong \mathbb{F}_q^{e_1}$ and $\mathbb{F}_{q^{e_2}} \cong \mathbb{F}_q^{e_2}$.

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The **concatenated code** $\mathcal{C} \subset (\mathbb{F}_q)^{e_1 n_1}$ is defined as $\mathcal{C} = Enc_1 \circ \psi \circ Enc_2((\mathbb{F}_{q^{e_2}})^{k_2})$, which is a $[e_1 n_1, e_2 k_2, d]_q$-linear code for some $d \geq d_1$.

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Observe that the encoding map for $\mathcal{C}$ is the composition:

$$\mathbb{F}_q^{e_2 k_2} \cong (\mathbb{F}_{q^{e_2}})^{k_2} \xrightarrow{Enc_2} (\mathbb{F}_{q^{e_2}})^{n_2} \cong (\mathbb{F}_{q^{e_1}})^{k_1} \xrightarrow{Enc_1} (\mathbb{F}_{q^{e_1}})^{n_1} \cong \mathbb{F}_q^{e_1 n_1}$$

Concatenation of Linear Codes (II)

As a particular case, $e_1 = 1$, $e_2 = k_1'$ and $\mathcal{C}_1$ consists of n_2 copies of a $[n_1', k_1', d_1']$ code $\mathcal{C}_1'$ (Each symbol in the outer codeword is encoded separately with $\mathcal{C}_1'$)
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$s_{11} \cdots s_{1k_1'} s_{21} \cdots s_{2k_1'} \cdots s_{k_2 1} \cdots s_{k_2 k_1'}$

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$$\begin{array}{cccccc} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_{k_2} & & \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{k_2} & \cdots & \mathbf{x}_{n_2} \end{array}$$

Outer codeword in $\mathcal{C}_2 \subset (\mathbb{F}_{q^{e_2}})^{n_2} \cong (\mathbb{F}_q^{k_1'})^{n_2}$

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$s_{11} \cdots s_{1k_1'} s_{21} \cdots s_{2k_1'} \cdots s_{k_2 1} \cdots s_{k_2 k_1'}$

$\mathbf{s}_1$	$\mathbf{s}_2$	$\cdots$	$\mathbf{s}_{k_2}$		
$\mathbf{x}_1$	$\mathbf{x}_2$	$\cdots$	$\mathbf{x}_{k_2}$	$\cdots$	$\mathbf{x}_{n_2}$
$\mathbf{y}_1$	$\mathbf{y}_2$	$\cdots$	$\mathbf{y}_{k_2}$	$\cdots$	$\mathbf{y}_{n_2}$

Inner codeword sequence in $(\mathcal{C}_1')^{n_2} \subset (\mathbb{F}_q^{n_1'})^{n_2}$

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$$
\begin{array}{llll}
s_{11} \cdots s_{1k_1'} & s_{21} \cdots s_{2k_1'} & \cdots s_{k_2 1} \cdots s_{k_2 k_1'} & \\
\mathbf{s}_1 & \mathbf{s}_2 & \cdots \quad \mathbf{s}_{k_2} & \\
\mathbf{x}_1 & \mathbf{x}_2 & \cdots \quad \mathbf{x}_{k_2} & \cdots \quad \mathbf{x}_{n_2} \\
\mathbf{y}_1 & \mathbf{y}_2 & \cdots \quad \mathbf{y}_{k_2} & \cdots \quad \mathbf{y}_{n_2} \\
y_{11} \cdots y_{1k_1'} \cdots y_{1n_1'} & y_{21} \cdots y_{2k_1'} \cdots y_{2n_1'} & \cdots y_{k_2 1} \cdots y_{k_2 k_1'} \cdots y_{k_2 n_1'} & \cdots y_{n_2 1} \cdots y_{n_2 n_1'}
\end{array}
$$

Final encoded sequence in $\mathbb{F}_q^{n_1' n_2}$

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\mathbf{s}_1 & & \mathbf{s}_2 & & \cdots & \mathbf{s}_{k_2} & & & & \\
\mathbf{x}_1 & & \mathbf{x}_2 & & \cdots & \mathbf{x}_{k_2} & & \cdots & \mathbf{x}_{n_2} & \\
\mathbf{y}_1 & & \mathbf{y}_2 & & \cdots & \mathbf{y}_{k_2} & & \cdots & \mathbf{y}_{n_2} & \\
y_{11} \cdots y_{1k_1'} & \cdots y_{1n_1'} & y_{21} \cdots y_{2k_1'} & \cdots y_{2n_1'} & \cdots & y_{k_2 1} \cdots y_{k_2 k_1'} & \cdots & y_{k_2 n_1'} & \cdots y_{n_2 1} \cdots y_{n_2 n_1'} &
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\mathbf{x}_1 & & \mathbf{x}_2 & & \cdots & \mathbf{x}_{k_2} & & \cdots & \mathbf{x}_{n_2} & \\
\mathbf{y}_1 & & \mathbf{y}_2 & & \cdots & \mathbf{y}_{k_2} & & \cdots & \mathbf{y}_{n_2} & \\
y_{11} \cdots y_{1k_1'} & \cdots & y_{1n_1'} & y_{21} \cdots y_{2k_1'} \cdots y_{2n_1'} & \cdots & y_{k_2 1} \cdots y_{k_2 k_1'} & \cdots & y_{k_2 n_1'} & \cdots & y_{n_2 1} \cdots y_{n_2 n_1'}
\end{array}
$$

The simplest example is taking $\mathcal{C}_1'$ the trivial code $\mathbb{F}_q^{k_1'}$.

This is just looking to a $[n, k, d]_{q^e}$ code as a $[en, ek, ed]_q$ code, which has the same rate, the same random error correction capability, but it can correct up to $\lfloor \frac{d_2 - 1}{4} \rfloor$ bursts of length e.

Example: Inner $15 \times [17, 9, 5]_2$, Outer $[15, 11, 5]_{2^9}$

Inner code: 15 copies of the previous example of $[17, 9, 5]_2$ BCH code

- it corrects up to 2 errors (each copy)
- it detects up to 4 errors (each copy)

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The concatenated code is a $[255, 99, d]_2$ code with $d \geq 25$.

- it corrects at least any 12 errors
- it can handle more errors if they follow some pattern...

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Decoding Strategy 1: Use the inner code to correct up to 2 errors per inner codeword. If error correction is impossible, mark the inner codeword as erasure.

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Bad inner decoding can only happen from 4 errors. All inner codeword with 2 or 3 errors are marked as erasures.
Bad outer decoding typically happens when there are more than 1 error in more than 4 inner codewords.

Concatenation of Linear Codes (III)

Another approach is using $e_1 = e_2 = 1$ and then $n_2 = k_1$. The encoding sequence is just

$$\mathbb{F}_q^{k_2} \xrightarrow{Enc_2} (\mathbb{F}_q)^{n_2} = (\mathbb{F}_q)^{k_1} \xrightarrow{Enc_1} (\mathbb{F}_q)^{n_1}$$

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For instance, we can use a $[28, 24, 5]_{2^8}$ shortened Reed-Solomon code as outer code and a $[32, 28, 5]_{2^8}$ shortened Reed-Solomon code as inner code.

Each layer adds four redundancy symbols (1024 bits) and can correct up to 2 errors or 4 erasures.

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. . . but the benefits come when concatenation os combined with interleaving. . .

Outline

Interleaving

Interleaving is just shuffling the symbols in the codewords in order to break the adjacency relations.

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For instance, the simplest way to interleave m codewords $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbf{F}_q^n$ is writing them as rows in a matrix and reading the matrix columnwise.

$$\begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \ldots & x_{mn} \end{pmatrix} \longrightarrow (x_{11}, x_{21}, \ldots, x_{m1}, \ldots, x_{mn})$$

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Then, an error burst of length $\leq r$ can only cause up to $\lceil \frac{r}{m} \rceil$ errors in the same codeword.

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If the source information is a data stream, interleaving can be based on delays.

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Starting from a sequence of codewords $\mathbf{x}_1, \mathbf{x}_2, \ldots \in \mathcal{C} \subset \mathbb{F}_q^n$, or the corresponding sequence of symbols in $\mathbb{F}_q$

$$\ldots x_{1,1} x_{1,2} \ldots x_{1,n}\ x_{2,1} x_{2,2} \ldots x_{2,n}\ x_{3,1} x_{3,2} \ldots x_{3,n} \ldots$$

we construct the interleaved sequence of symbols in $\mathbb{F}_q$

$$\ldots x_{n,1} x_{n-1,2} \ldots x_{1,n}\ x_{n+1,1} x_{n,2} \ldots x_{2,n}\ x_{n+2,1} x_{n+1,2} \ldots x_{3,n} \ldots$$

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An error burst of length $n+1$ is deinterleaved into single errors affecting $n+1$ codewords, which will be easily corrected.

Example of Cross Interleaving

For example, consider $n = 4$, the codeword $\mathbf{x}_i$ is represented as (a_i, b_i, c_i, d_i), and undefined symbols are set to 0 (this only applies to the start and end of the stream).

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$$\begin{array}{l} \cdots 00 a_1 a_2 a_3 a_4 a_5 \cdots \\ \cdots 00 b_1 b_2 b_3 b_4 b_5 \cdots \\ \cdots 00 c_1 c_2 c_3 c_4 c_5 \cdots \\ \cdots 00 d_1 d_2 d_3 d_4 d_5 \cdots \end{array} \rightarrow \quad a_1 000 a_2 b_1 00 a_3 b_2 c_1 0 a_4 b_3 c_2 d_1 a_5 b_4 c_3 d_2 \cdots$$

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An error burst of length 5 is transformed into single errors in 5 consecutive codewords.

Concatenation and Interleaving (I)

In practice, interleaving and concatenation are combined to achieve resilience to long error bursts.

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Consider now cross interleaving after encoding with a $[6, 4, 3]$ inner code.

$$
\begin{array}{l}
a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8\, a_9 \cdots \\
b_1\, b_2\, b_3\, b_4\, b_5\, b_6\, b_7\, b_8\, b_9 \cdots \\
c_1\, c_2\, c_3\, c_4\, c_5\, c_6\, c_7\, c_8\, c_9 \cdots \\
d_1\, d_2\, d_3\, d_4\, d_5\, d_6\, d_7\, d_8\, d_9 \cdots \\
e_1\, e_2\, e_3\, e_4\, e_5\, e_6\, e_7\, e_8\, e_9 \cdots \\
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c_1\, c_2\, c_3\, c_4\, c_5\, c_6\, c_7\, c_8\, c_9 \cdots \\
d_1\, d_2\, d_3\, d_4\, d_5\, d_6\, d_7\, d_8\, d_9 \cdots \\
e_1\, e_2\, e_3\, e_4\, e_5\, e_6\, e_7\, e_8\, e_9 \cdots \\
f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7\ f_8\ f_9 \cdots \\
\\
1\ 2\ 2\ 2\ 3\ 2\ 1\ 0\ 0 \cdots
\end{array}
$$

An error burst of length 13 produces up to three errors in the same inner codeword.

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$$\begin{array}{l} a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8\, a_9 \cdots \\ b_1\, b_2\, b_3\, b_4\, b_5\, b_6\, b_7\, b_8\, b_9 \cdots \\ c_1\, c_2\, c_3\, c_4\, c_5\, c_6\, c_7\, c_8\, c_9 \cdots \\ d_1\, d_2\, d_3\, d_4\, d_5\, d_6\, d_7\, d_8\, d_9 \cdots \\ e_1\, e_2\, e_3\, e_4\, e_5\, e_6\, e_7\, e_8\, e_9 \cdots \\ f_1\ \ f_2\ \ f_3\ \ f_4\ \ f_5\ \ f_6\ \ f_7\ \ f_8\ \ f_9 \cdots \\ \\ 1\ \ 2\ \ 2\ \ 2\ \ 3\ \ 2\ \ 1\ \ 0\ \ 0 \cdots \end{array} \quad \rightarrow \quad \begin{array}{l} a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8\, a_9 \cdots \\ b_1\, b_2\, b_3\, b_4\, b_5\, b_6\, b_7\, b_8\, b_9 \cdots \\ c_1\, c_2\, c_3\, c_4\, c_5\, c_6\, c_7\, c_8\, c_9 \cdots \\ d_1\, d_2\, d_3\, d_4\, d_5\, d_6\, d_7\, d_8\, d_9 \cdots \end{array}$$

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Concatenation and Interleaving (II)

With another layer of interleaving (e.g., double delay cross interleaving) after encoding with an outer code of minimum distance 4 (e.g., the repetition code $[4, 1, 4]$) all errors and erasures are successfully corrected!

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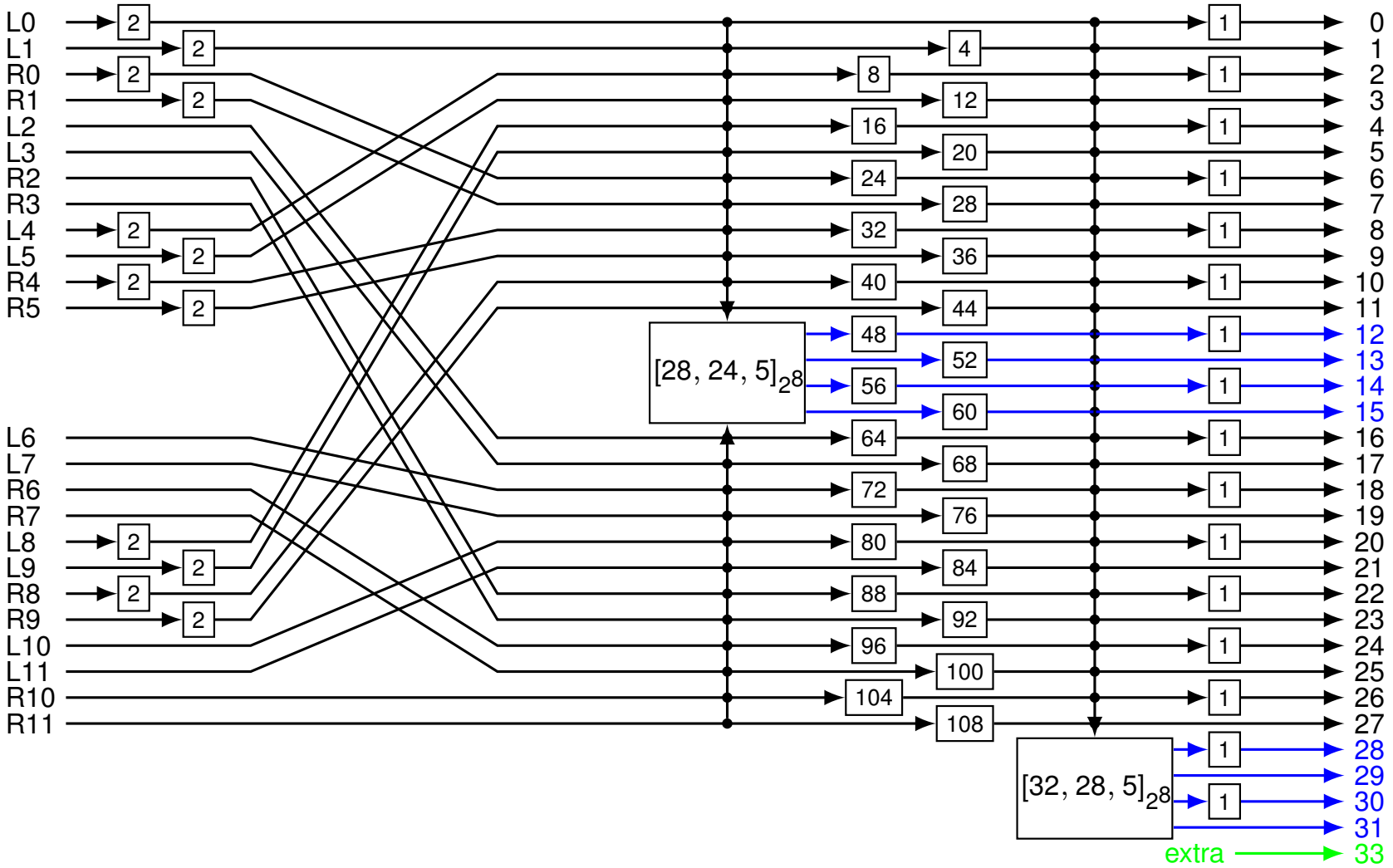
$$
\begin{array}{llllllllllll}
\cdots & a_{-4} & a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \cdots \\
\cdots & b_{-2} & b_{-1} & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \cdots \\
\cdots & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} \cdots \\
\cdots & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 & d_9 & c_{10} & c_{11} & c_{12} \cdots \\
\cdots & 1 & 1 & 2 & 1 & 3 & 1 & 3 & 1 & 2 & 0 & 1 \cdots \quad \text{(erasures)} \\
\cdots & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \cdots \quad \text{(errors)}
\end{array}
$$

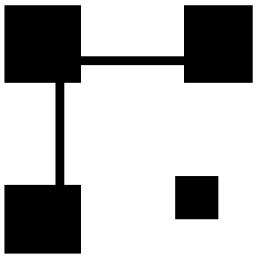
Outline

Compact Disk Interleaving Diagram

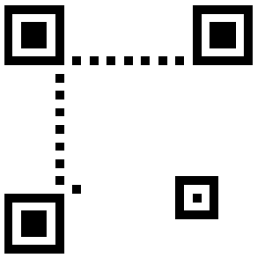


QR Barcodes Interleaving (I)

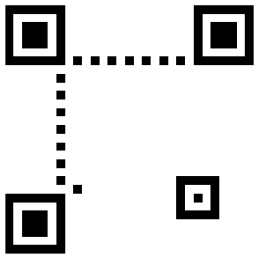
Reserved Areas



Alignement Marks



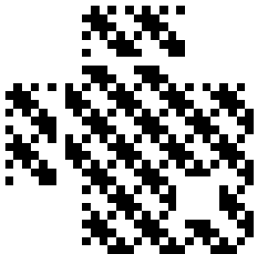
No Version Info Added



QR Unmasked

with Format Info

Mask



QR Code (w/o border)

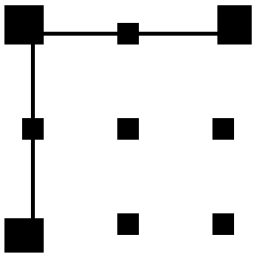
QR Code (Final)

Interleaving Map

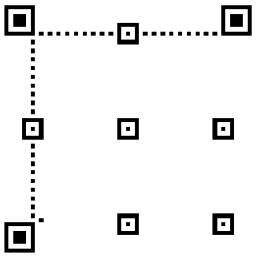
QR Code version number: 3
QR Code size (w/o border): 29×29
Available space: 70 bytes
Error correction level: H
Using field $\mathbb{F}_2[X]/(X^8 + X^4 + X^3 + X^2 + 1)$ for data encoding
Using 2 blocks of shortened RS code $[35, 13, 23]_{256}$
Generating polynomial coeffs. (in hexadecimal)
01 59 B3 83 B0 B6 F4 13 BD 45 28 1C 89 1D 7B 43 FD 56 DA E6 1A 91 F5
Total raw capacity: 26 bytes
Total parity bytes: 44
Data preencoded in 'binary' format
Encoded blocks (in hexadecimal):
40 E4 86 56 C6 C6 F2 05 76 F7 26 C6 42 D8 BB 91 43 DE F7 47 6F AE 88 42 42 A4 4B A5 5C E5 25 81 3C 55 08
12 12 10 EC 11 EC 11 EC 11 EC 11 EC 11 F6 7B 07 19 07 03 1E 79 F5 84 9C 12 31 DB 03 D3 FA 66 C8 A4 86 AE
Interleaved blocks (in hexadecimal):
40 12 E4 12 86 10 56 EC C6 11 C6 EC F2 11 05 EC 76 11 F7 EC 26 11 C6 EC 42 11 D8 F6 BB 7B 91 07 43 19 DE 07 F7 03 47 1E 6F 79 AE F5 88 84 42 9C 42 12 A4 31 4B DB A5 03 5C D3 E5 FA 25 66 81 C8 3C A4 55 86 08 AE

QR Barcodes Interleaving (II)

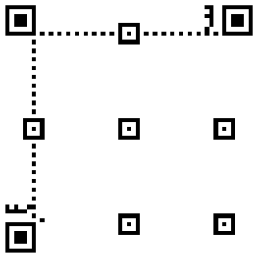
Reserved Areas



Alignement Marks



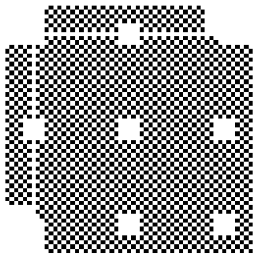
Version Info Added



QR Unmasked

with Format Info

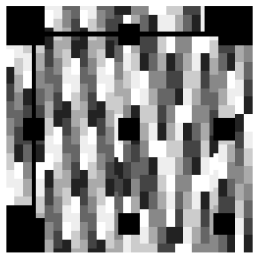
Mask



QR Code (w/o border)

QR Code (Final)

Interleaving Map



QR Code version number: 10
QR Code size (w/o border): 57×57
Available space: 346 bytes
Error correction level: H
Using field $\mathbb{F}_2[X]/(X^8 + X^4 + X^3 + X^2 + 1)$ for data encoding
Using 6 blocks of shortened RS code $[43, 15, 29]_{256}$ and 2 blocks of $[44, 16, 29]_{256}$
Generating polynomial coeffs. (in hexadecimal)
01 FC 09 1C 0D 12 FB D0 96 67 AE 64 29 A7 0C F7 38 75 77 E9 7F B5 64 79 93 B0 4A 3A C5
Total raw capacity: 122 bytes
Total parity bytes: 224
Data preencoded in 'binary' format
Encoded blocks (in hexadecimal):
40 00 E4 86 56 C6 C6 F2 05 76 F7 26 C6 42...
12 10 EC 11 EC 11 EC 11 EC 11 EC 11 EC 11...
11 EC 11 EC 11 EC 11 EC 11 EC 11 EC 11 EC...
EC 11 EC 11 EC 11 EC 11 EC 11 EC 11 EC 11...
11 EC 11 EC 11 EC 11 EC 11 EC 11 EC 11 EC...
EC 11 EC 11 EC 11 EC 11 EC 11 EC 11 EC 11...
11 EC 11 EC 11 EC 11 EC 11 EC 11 EC 11 EC...
11 EC 11 EC 11 EC 11 EC 11 EC 11 EC 11 EC...
Interleaved blocks (in hexadecimal):
40 12 11 EC 11 EC 11 11 00 10 EC 11 EC 11 EC EC
E4 EC 11 EC 11 EC 11 11 86 11 EC 11 EC 11 EC EC
56 EC 11 EC 11 EC 11 11 C6 11 EC 11 EC 11 EC EC
C6 EC 11 EC 11 EC 11 11 F2 11 EC 11 EC 11 EC EC
05 EC 11 EC 11 EC 11 11 76 11 EC...

Real Systems (I)

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Extra error correction can be performed if the source is redundant (and not compressed). E.g., interpolation of missing samples in digital audio produces reasonable results.

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There can be physical limitations that forces structuring the encoded data in a particular way.

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There can be physical limitations that forces structuring the encoded data in a particular way.

The information stored in a Compact Disk can be modelled by a binary string in which ones are always separated by a number of zeros between 2 and 10. Thus, codewords must be reencoded with a special code (EFM or Eight-To-Fourteen Modulation) to fulfil these requirements.

Eight-To-Fourteen Modulation

It is a nonlinear binary code of length 14 and size 257 (not intended for error correction).

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The code is

00000000100000 00000000100001 00000000100010 00000000100100 00000001000000 00000001000001

00000001000010 00000001000100 00000001001000 00000001001001 00000010000000 00000010000001

00000010000010 00000010000100 00000010001000 00000010001001 ... 10010010010010

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Only 256 codewords are used (to map every element in $\mathbb{F}_{256}$ to the code).

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The code is

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00000001000010 00000001000100 00000001001000 00000001001001 00000010000000 00000010000001

00000010000010 00000010000100 00000010001000 00000010001001 ... 10010010010010

Only 256 codewords are used (to map every element in $\mathbb{F}_{256}$ to the code).

Codewords are separated by three extra binary digits (to avoid breaking the rules about the number of separating zeros).

Codes and Cryptography

Jorge L. Villar

MAMME, Fall 2015

END OF PART VIII