

Codes and Cryptography

Jorge L. Villar

MAMME, Fall 2015

PART VI

Outline

- 1 Linear Codes
- 2 Perfect Linear Codes
- 3 Linear MDS Codes

Linear Codes

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- $\mathbf{0} = (0, \dots, 0)$ is always a codeword.
- The minimum distance is now

$$d_{\mathcal{C}} = \min_{\mathbf{x}, \mathbf{y} \in \mathcal{C}, \mathbf{x} \neq \mathbf{y}} d(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{x} \in \mathcal{C}, \mathbf{x} \neq \mathbf{0}} d(\mathbf{x}, \mathbf{0})$$

The distance $d(\mathbf{x}, \mathbf{0})$ is the number of non-zero components, also called the **weight** of \mathbf{x} . Thus, $d_{\mathcal{C}}$ is the minimum positive weight.

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We call the code a $[n, k, d]_q$ -linear code.

Generating Matrix

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In a previous example in \mathbb{F}_3^4 the linear code $\mathcal{C} = \{0000, 0122, 0211, 1012, 1101, 1220, 2021, 2110, 2202\}$ can be generated from $\{0122, 1220\}$, defining the generating matrix

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Encoding is simply multiplying by G :
For any $\mathbf{s} \in \mathbb{F}_q^k$, $\text{Enc}(\mathbf{s}) = \mathbf{s}G$

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In the example,

$$G = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \end{array} \right)$$

Then $\text{Enc}((s_1, s_2)) = (s_1, s_2, s_1 + 2s_2, 2s_1 + 2s_2)$

“Parity Check” Matrix

For a $[n, k, d]_q$ -linear code \mathcal{C} with generating matrix G , we call “**parity-check**” matrix, or better the error-checking matrix, to any $(n - k) \times n$ full-rank matrix H such that

$$GH^T = \mathbf{0}^{k \times (n-k)}$$

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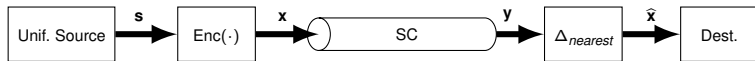
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Curiously, in the example we can just take $H = G$ as
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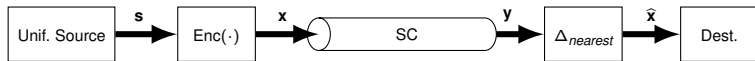
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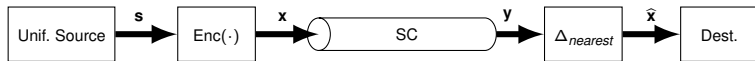
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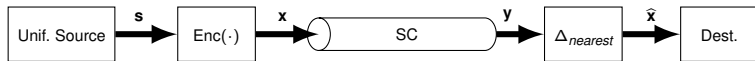
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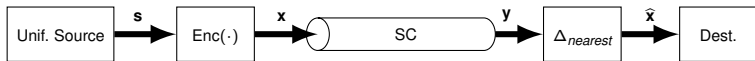
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It requires storage $((\binom{n}{1} + \dots + \binom{n}{t})) n \log_2 q$.

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- $(\mathcal{C}^\perp)^\perp = \mathcal{C}$
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Notice that a linear code can only be self-dual if $n = 2k$.

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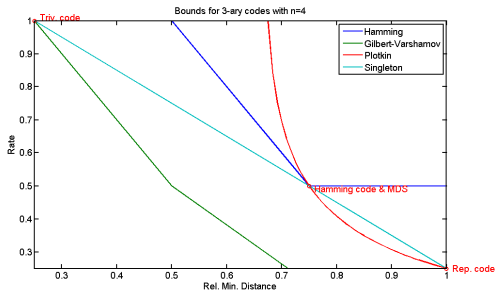
They can correct only one error

Example of a Hamming $[4, 2, 3]_3$ Code

The previously used example: $q = 3$, $m = 2$, $n = 4$, $k = 2$

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

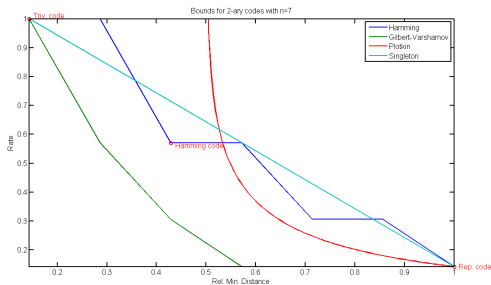
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Example of a Hamming $[7, 4, 3]_2$ Code

$$q = 2, m = 3, n = 7, k = 4$$

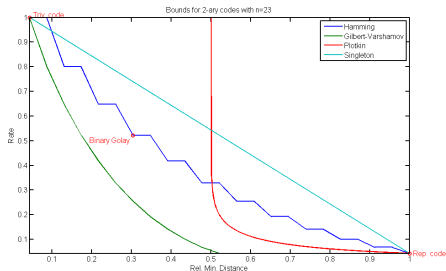
$$H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$



Binary Golay Code

$[23, 12, 7]_2$ -linear perfect code with generator matrix

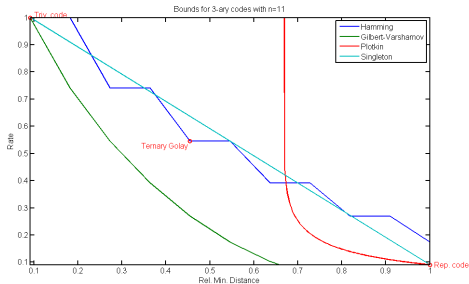
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Ternary Golay Code

$[11, 6, 5]_3$ -linear perfect code

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 2 \end{pmatrix}$$



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- For a linear MDS code, if $1 < k < q$ and q is prime, then $n \leq q + 1$
E.g., the (extended) Reed-Solomon code

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The image $\mathcal{C} = ev(\mathbb{F}_q[X]_{\leq k-1})$ is the Reed-Solomon code, which is a $[q, k, q - k + 1]_q$ -linear MDS code.

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The extended Reed-Solomon code is proven to be MDS in a similar way.

Reed-Solomon Codes (III)

The generating matrix when the polynomials are evaluated at $\{0, 1, \alpha, \dots, \alpha^{n-2}\}$ is the Vandermonde matrix:

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & \alpha & \alpha^2 & \dots & \alpha^{n-2} \\ 0 & 1 & \alpha^2 & \alpha^4 & \dots & \alpha^{2(n-2)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & \alpha^{k-1} & \alpha^{2(k-1)} & \dots & \alpha^{(k-1)(n-2)} \end{pmatrix}$$

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$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & \alpha & \alpha^2 & \dots & \alpha^{n-2} \\ 0 & 1 & \alpha^2 & \alpha^4 & \dots & \alpha^{2(n-2)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & \alpha^{k-1} & \alpha^{2(k-1)} & \dots & \alpha^{(k-1)(n-2)} \end{pmatrix}$$

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The columns can be reordered without affecting the code properties

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$$(h(\xi_1), \dots, h(\xi_k)) \mapsto (h(\xi_1), \dots, h(\xi_n))$$

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The map is well-defined, because $(h(\xi_1), \dots, h(\xi_k))$ uniquely determines a polynomial $h \in \mathbb{F}_q[X]$ of degree $\leq k - 1$.

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Finding a primitive element in \mathbb{F}_7 :

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Building the generating matrix:

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 6 & 4 & 5 & 0 \\ 0 & 1 & 2 & 4 & 1 & 2 & 4 & 0 \\ 0 & 1 & 6 & 1 & 6 & 1 & 6 & 1 \end{pmatrix}$$

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Turning it into systematic by Gauss elimination:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 4 & 6 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 4 & 1 & 4 \\ 0 & 0 & 1 & 0 & 6 & 4 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 & 1 & 1 & 3 \end{pmatrix}$$

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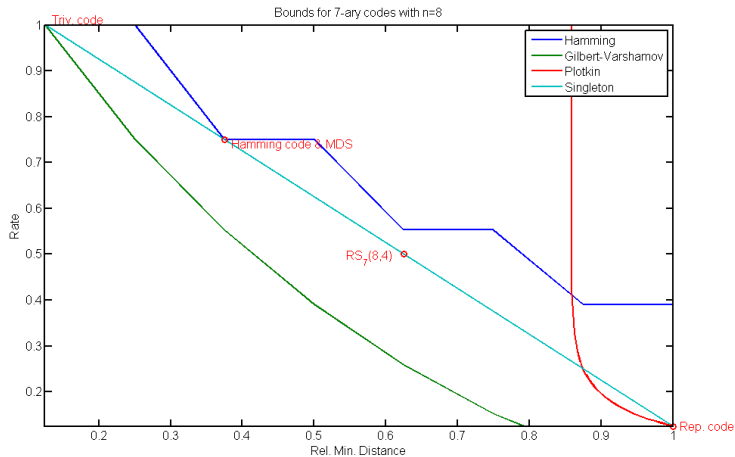
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The code is selfdual! ... Choosing $H = G$

The $[8, 4, 5]_7$ Reed-Solomon code:



Codes and Cryptography

Jorge L. Villar

MAMME, Fall 2015

END OF PART VI